343H: Honors AI

Lecture 8

Probability

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Slides courtesy of Dan Klein, UC Berkeley

Unless otherwise noted

Announcements

- Blackboard: view your grades and feedback on assignments.
- Typically can expect Pset grades by 1 week after deadline.

Today

- Last time: Games with uncertainty
  - Expectimax search
  - Mixed layer and multi-agent games
  - Defining utilities
  - Rational preferences
  - Human rationality, risk, and money

- Today: Probability

Recall: Rational Preferences

- Preferences of a rational agent must obey constraints.
  - The axioms of rationality:

Orderability

$$(A \succ B) \lor (B \succ A) \lor (A \sim B)$$

Transitivity

$$(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A; \; 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; \; 1-p, B] \succeq [q, A; \; 1-q, B])$$

- Theorem: Rational preferences imply behavior describable as maximization of expected utility

Recall: MEU Principle

- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function U such that:

$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

$$U([p_1, S_1;\ \ldots\ ;\ p_n, S_n]) = \sum_i p_i U(S_i)$$

  - i.e., values assigned by U preserve preferences of both prizes and lotteries!

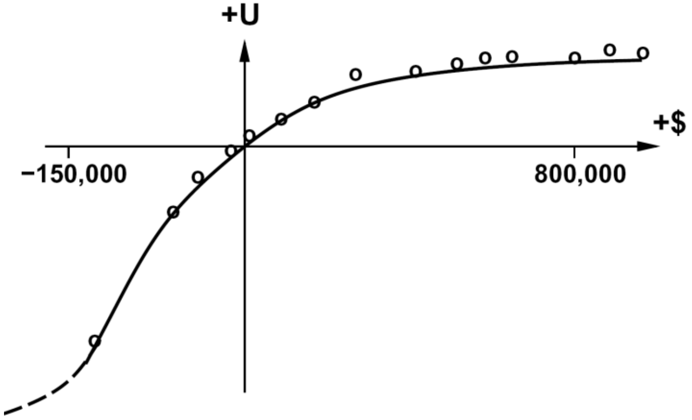
- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
    - E.g., a lookup table for perfect tictactoe, reflex vacuum cleaner

Recall: Money

- Money <u>does not</u> behave as a utility function, but we can talk about the utility of having money (or being in debt)

- Given a lottery L = [p, $X; (1-p), $Y]
  - The expected monetary value EMV(L) is p*X + (1-p)*Y
  - U(L) = p*U($X) + (1-p)*U($Y)
  - Typically, U(L) < U(EMV(L)): why?

  - In this sense, people are risk-averse
  - When deep in debt, we are risk-prone



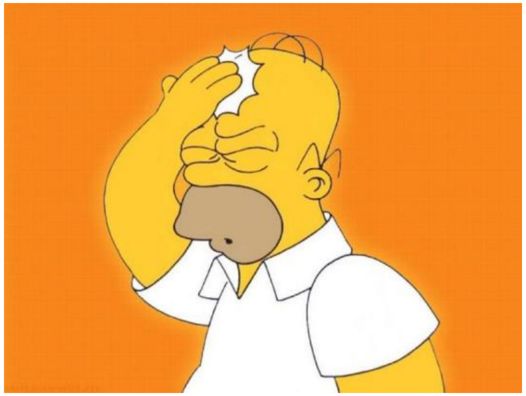
Example: Insurance

- Consider the lottery [0.5,$1000; 0.5,$0]
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!

Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8,$4k; 0.2,$0]
  - B: [1.0,$3k; 0.0,$0]

  - C: [0.2,$4k; 0.8,$0]
  - D: [0.25,$3k; 0.75,$0]



- Most people prefer B > A, C > D

- But if U($0) = 0, then
  - B > A $\Rightarrow$ U($3k) > 0.8 U($4k)
  - C > D $\Rightarrow$ 0.8 U($4k) > U($3k)

Today

- Last time: Games with uncertainty
  - Expectimax search
  - Mixed layer and multi-agent games
  - Defining utilities
  - Rational preferences
  - Human rationality, risk, and money

- Today: Probability

Need for probability

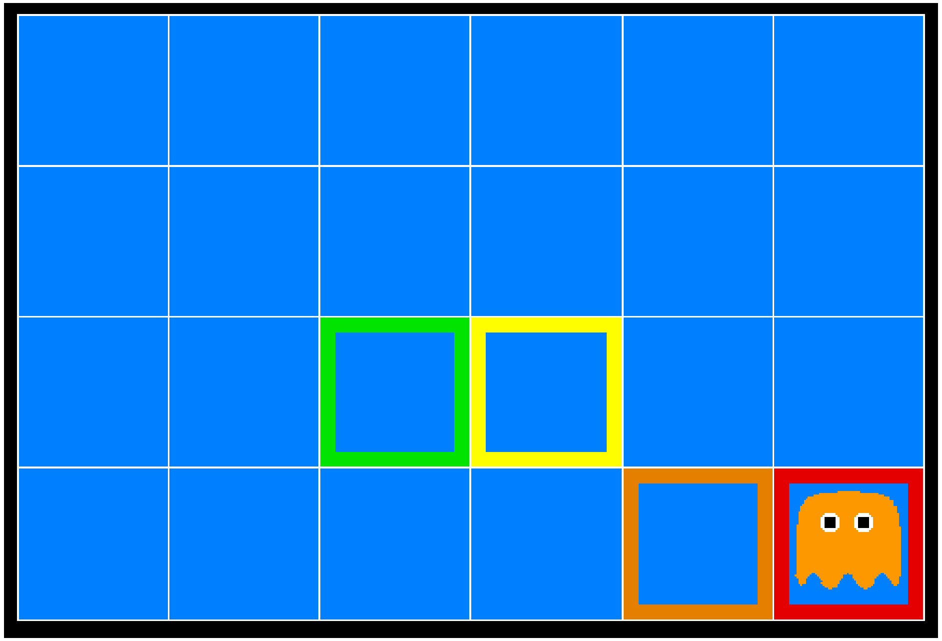
- Search and planning
- Probabilistic reasoning (Part II of course)
  - Diagnosis
  - Speech recognition
  - Tracking objects
  - Robot mapping
  - Genetics
  - Error correcting codes
  - …lots more!
- Machine learning (Part III of course)

Topics

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Inference
  - Independence
- You'll need all this stuff A LOT in subsequent weeks, so make sure you go over it now!

Inference in Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green



- Sensors are noisy, but we know P(Color | Distance)

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

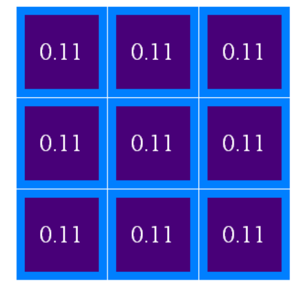
Inference in Ghostbusters

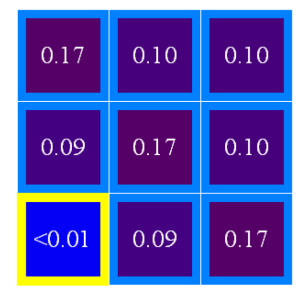
0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

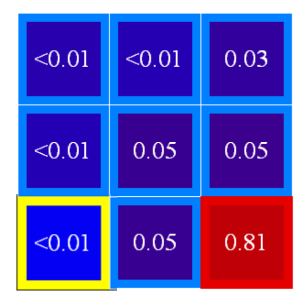
0.17	0.10	0.10
0.09	0.17	0.10
<0.01	0.09	0.17

Uncertainty

- General situation:
  - **Observed variables (evidence)**: Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables**: Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model**: Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge







Random Variables

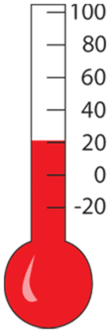
- A **random variable** is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - D = How long will UT delay for winter weather?
  - L = Where is the ghost?

- We denote random variables with capital letters

- Random variables have **domains**
  - R in {true, false} (sometimes write as {+r, $\neg$r})
  - D in [0, 8)
  - L in possible locations, maybe {(0,0), (0,1), …}

Probability Distributions

- Unobserved random variables have distributions

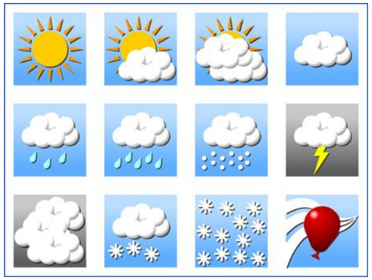


$P(T)$

T	P
warm	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0



- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number

$$P(W = rain) = 0.1 \qquad P(rain) = 0.1$$

- Must have: $\forall x \, P(x) \geq 0$ $\qquad \sum_x P(x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Size of distribution if n variables with domain sizes d?
  - Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- For all but the smallest distributions, impractical to write out

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like P(T=hot)

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz

1. P(+x, +y)?

2. P(+x)?

3. P(-y OR +x) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- *Marginal distributions* are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t,s)$$

$$P(s) = \sum_t P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: marginal distributions

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	
-x	

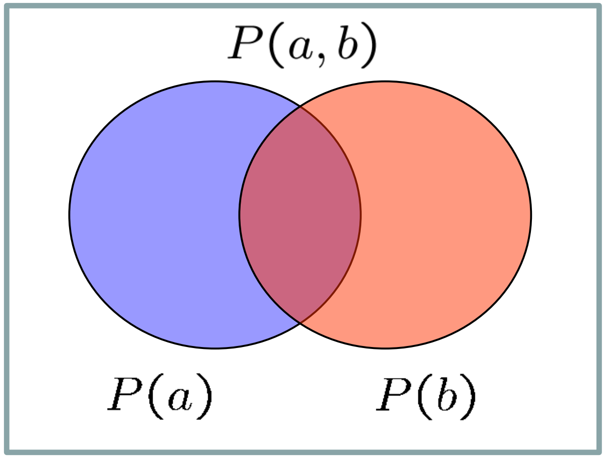
$P(Y)$

Y	P
+y	
-y	

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=r|T=c) = ???$$

Quiz: conditional probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$$P(W|T)$$

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

$$P(W|T = cold)$$

W	P
sun	0.4
rain	0.6

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Computing conditional probabilities

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$
\begin{aligned}
P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\
&= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\
&= \frac{0.2}{0.2+0.3} = 0.4
\end{aligned}
$$

$$
\begin{aligned}
P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\
&= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\
&= \frac{0.3}{0.2+0.3} = 0.6
\end{aligned}
$$

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

Normalization Trick

- A trick to get a whole conditional distribution at once:
  1. Select the joint probabilities matching the evidence
  2. Normalize the selection (make it sum to one)

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Select

P(c,W)

T	R	P
cold	sun	0.2
cold	rain	0.3

0.5

Normalize

P(W | T=c)

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is P(evidence)! (P(c) here)

$$P(x_1|x_2) = \frac{P(x_1,x_2)}{P(x_2)} = \frac{P(x_1,x_2)}{\sum_{x_1} P(x_1,x_2)}$$

Quiz: normalization trick

- P(X | Y=-y) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence →

NORMALIZE the selection (make it sum to one) →

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

Inference by Enumeration

- P(sun)?

- P(sun | winter)?

- P(sun | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \dots H_r$

$X_1, X_2, \dots X_n$

All variables

- We want: $P(Q|e_1 \dots e_k)$

1. Select the entries consistent with the evidence
2. Sum out H to get joint of Query and evidence:

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

3. Normalize

$$Z = \sum_q P(Q, e_1, \dots, e_k)$$

$$P(Q|e_1, \dots, e_k) = \frac{1}{Z} \sum_q P(Q, e_1, \dots, e_k)$$

** Works fine with multiple query variables, too*

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad \Longleftrightarrow \quad P(x,y) = P(x|y)P(y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

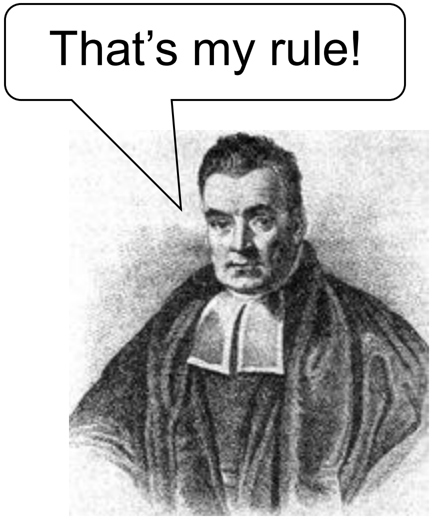
$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

Bayes’ Rule

- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$



- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we’ll see later

- In the running for most important AI equation!

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{Cause}|\text{Effect}) = \frac{P(\text{Effect}|\text{Cause})P(\text{Cause})}{P(\text{Effect})}$$

- Example:
  - m is meningitis, s is stiff neck

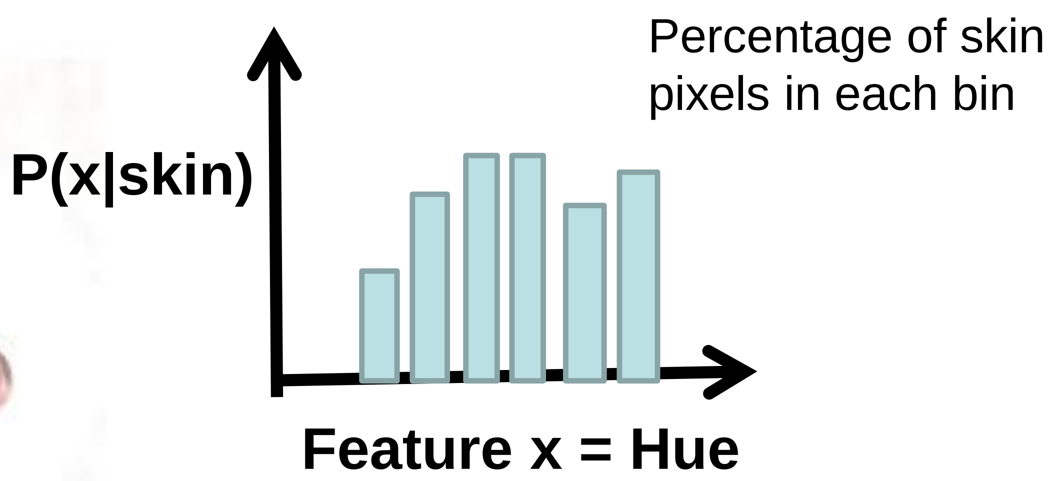
$$\left.\begin{aligned} P(s|m) &= 0.8 \\ P(m) &= 0.0001 \\ P(s) &= 0.1 \end{aligned}\right\} \text{Example givens}$$

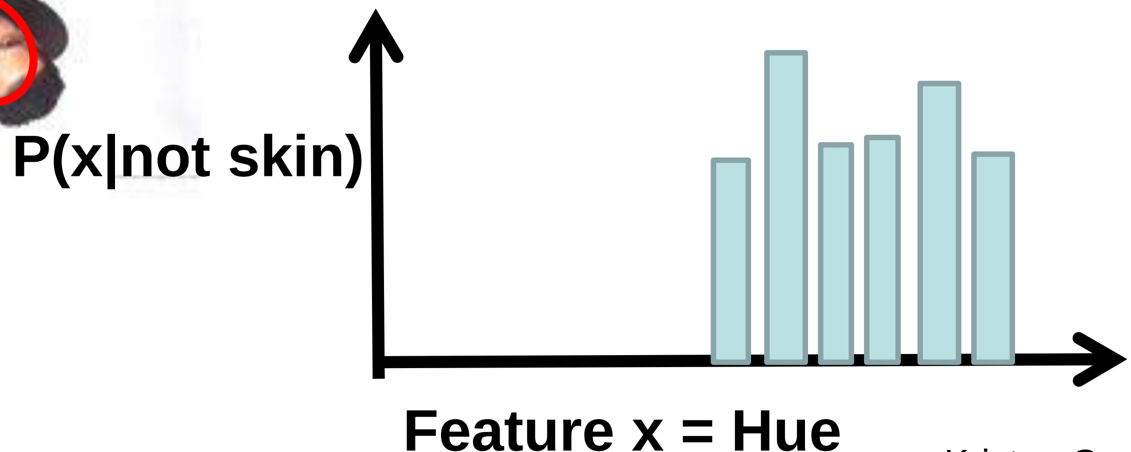
$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out! Why?

Example: learning skin colors

- We can represent a class-conditional density using a histogram (a "non-parametric" distribution)





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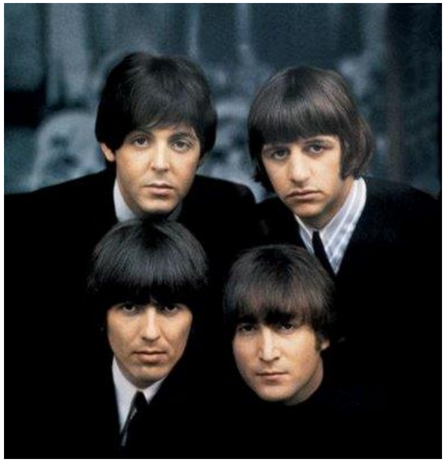
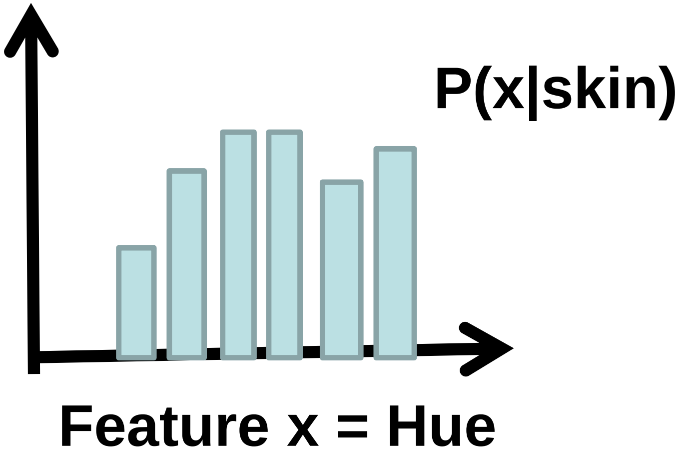
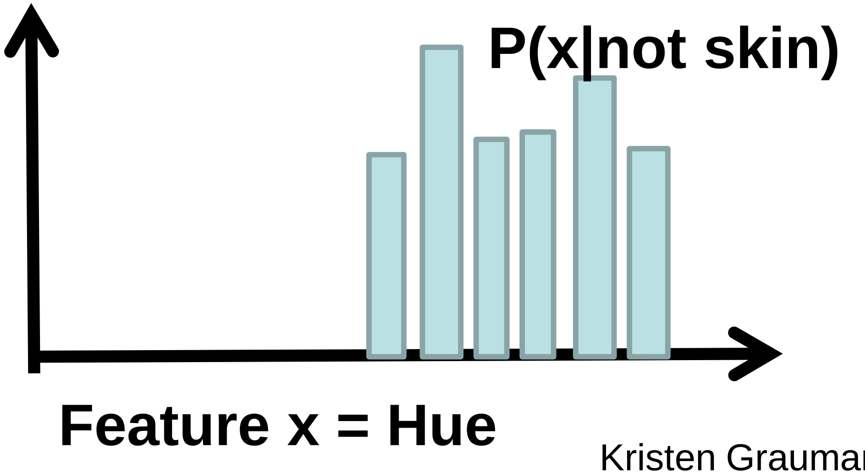
Example: learning skin colors

- We can represent a class-conditional density using a histogram (a “non-parametric” distribution)







Now we get a new image, and want to label each pixel as skin or non-skin.

What’s the probability we care about to do skin detection?

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Example: learning skin colors

$$P(skin \mid x) = \frac{P(x \mid skin)P(skin)}{P(x)}$$

$$P(skin \mid x) \,\alpha\, P(x \mid skin)P(skin)$$

Where might the prior come from?

Example: learning skin colors

Now for every pixel in a new image, we can estimate probability that it is generated by skin.

Brighter pixels → higher probability of being skin

Classify pixels based on these probabilities

- if $p(\text{skin}|\boldsymbol{x}) > \theta$, classify as skin
- if $p(\text{skin}|\boldsymbol{x}) < \theta$, classify as not skin

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Quiz: Bayes' Rule

- What is P(W | dry) ?

$P(W)$

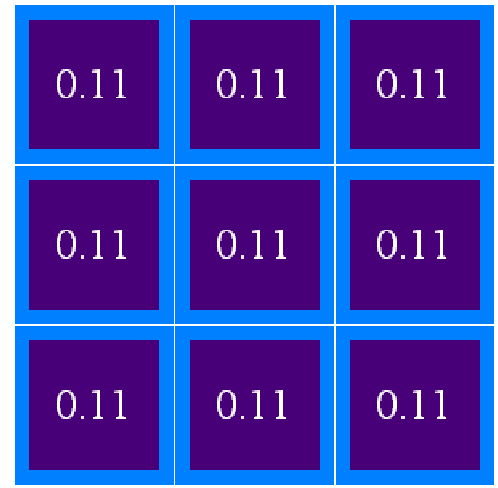
R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

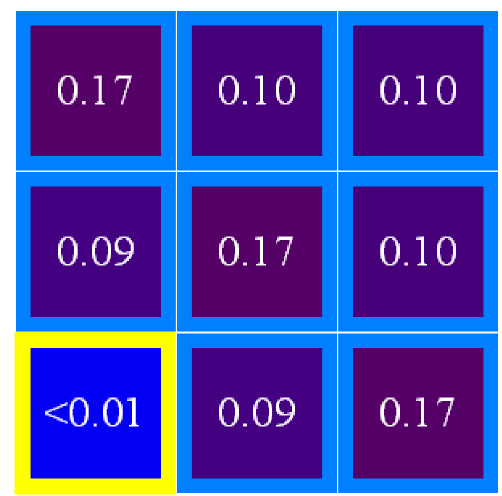
Ghostbusters, Revisited

- Let’s say we have two distributions:
  - Prior distribution over ghost location: P(G)
    - Let’s say this is uniform
  - Sensor reading model: P(R | G)
    - Given: we know what our sensors do
    - R = reading color measured at (1,1)
    - E.g. P(R = yellow | G=(1,1)) = 0.1



- We can calculate the posterior distribution P(G|r) over ghost locations given a reading using Bayes’ rule:

$$P(g|r) \propto P(r|g)P(g)$$



Summary

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Inference

- Next time:
  - Independence
  - Bayesian Networks