# Yet another note about termination.

I had never thought that I would have to write this note, but apparently I have to. In a paper I had written:

> "For each repetitive process we must have a monotonicity argument on which to base our proof of termination."

To my utter amazement, the Editor of the journal to which it was submitted, expressed in his letter seven lines of severe doubt about the above statement, ending with: "Perhaps it is true, but it is a rather sweeping claim." I could only conclude that the need for a monotonicity argument for termination proofs is not the common knowledge I supposed it to be, and that the need, even when stated, is not intuitively obvious to everybody.

Because perhaps my Editor's hesitation was caused by the nondeterminacy that also played a role in that paper, let us consider the general --and in general nondeterministic-- repetitive construct DO:

$$\underline{\text{do}}\ \ B1 \rightarrow S1\ [\!]\ \ldots\ [\!]\ Bn \rightarrow Sn\ \ \underline{\text{od}} \quad .$$

On my recent trip through the USSR, Tony Hoare, who is presently more operationally inclined than I, described it as

$$\{\underline{\text{if}}\ B1 \rightarrow S1\ [\!]\ \ldots\ [\!]\ Bn \rightarrow Sn\ \underline{\text{fi}}\}^* \quad \text{or} \quad \{IF\}^*$$

where with $\{\ldots\}^*$ he meant "as many successive executions as possible", where the execution of IF is regarded when all the guards are false (non BB). A termination proof for a given state x means that after a finite number of steps --i.e. applications of IF -- the state non BB is reached. The actual number of steps may not be determined by x , e.g. $\underline{\text{do}}\ y \geq 2 \rightarrow y:= y - 2\ [\!]\ y \geq 1 \rightarrow y:= y - 1\ \underline{\text{od}}$ with initially $y \geq 2$ ; guaranteed termination for the initial state x , however, means that the maximum number of steps needed is finite. Denoting this maximum number by mn(x) , termination is guaranteed in all points x where mn(x) is finite. Denoting the transformation of the state as effectuated by IF symbolically by x:= F(x) --where F may be a partial function: F(x) need not be defined for states x , in which BB does not hold-- it is clear that with the above meaning of mn we must have

$$(mn(x) > 0) \Rightarrow (mn(F(x)) \leq mn(x) - 1) \quad .$$

Note. If F(x) is a single valued function, the program is deterministic and the maximum number of steps needed is also the actual number of steps needed;

and then we have

$$(mn(x) > 0) \Rightarrow (mn(F(x)) = mn(x) - 1) \quad . \qquad \text{(End of note.)}$$

Hence, for all initial states in which termination is guaranteed to occur, the finite function $mn(x)$ which decreases monotonically by at least one at each application of $x:= F(x)$ exists; conversely: each proof of termination boils down to a proof of the existence of such a monotonically decreasing function.

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Because my challenged claim is one about all possible arguments for termination, a less operational approach that is directly based on the axiomatic definition of the repetitive construct might be appreciated.

For the repetitive construct DO the predicate transformation --see [1], page 35-- is given in terms of the predicate transformation of the corresponding alternative construct IF by

$$\begin{aligned} &H_0(R) = R \ \underline{\text{and}}\ \underline{\text{non}}\ BB \\ &H_{k+1}(R) = wp(IF,\ H_k(R))\ \underline{\text{or}}\ H_0(R) \qquad\qquad (2) \\ &wp(DO,\ R) = (\underline{E}\ k:\ k \geq 0:\ H_k(R)) \quad . \end{aligned}$$

From (2) it follows that for $i < j$ we have $H_i(R) \Rightarrow H_j(R)$ for all states and all post-conditions $R$. Proving termination means showing that a point $x$ satisfies $wp(DO, T)$, i.e. that there exists a value $t(x)$ such that $H_k(T)$ holds there for all values of $k$ satisfying $k \geq t(x)$. As $x$ satisfies an invariant relation, $P$ say, it suffices to prove that

$$(P\ \underline{\text{and}}\ t \leq k) \Rightarrow H_k(T) \qquad \text{for all states } x \text{ .}$$

But this is the same formula as on the top of [1], page 42. In view of the fact that (2) <u>defines</u> $H_{k+1}$ in terms of $H_k$, mathematical induction over $k$ is <u>by definition</u> the only available pattern of reasoning. The argument is carried out in [1], pages 41/42; it shows quite clearly how for the transition from $k$ to $k+1$ the monotonicity assumption about IF, viz.

$$(P\ \underline{\text{and}}\ BB\ \underline{\text{and}}\ t \leq t0+1) \Rightarrow wp(IF,\ t \leq t0)$$

is essentially needed.

[1] Dijkstra, Edsger W., "A Discipline of Programming". Prentice-Hall, 1976

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