An experiment in mathematical exposition.

Many people feel attracted to the implication on account of the simplicity of the associated inference rules

$$\frac{\begin{array}{c}A \Rightarrow B\\ B \Rightarrow C\end{array}}{A \Rightarrow C} \qquad (1)$$

$$\frac{\begin{array}{c}A \Rightarrow B\\ C \Rightarrow D\end{array}}{A \wedge C \Rightarrow B \wedge D} \qquad (2)$$

$$\frac{\begin{array}{c}A \Rightarrow B\\ C \Rightarrow D\end{array}}{A \vee C \Rightarrow B \vee D} \qquad (3)$$

The transitivity of (1), and the symmetry of (2) and of (3) are clearly appealing. Rule (1), however, is a direct consequence of, and rules (2) and (3) are merely two different transcriptions of the same

$$\frac{\begin{array}{c}A \vee B\\ C \vee D\end{array}}{A \vee C \vee (B \wedge D)} \qquad (4)$$

,

a rule, which —on account of the symmetry of the disjunction— can be applied in <u>four</u> different ways to the two given antecedents. Rules (2) and (3) give only two of the four. Rule (1) emerges as the special case

$$\begin{array}{l} A \vee B \\ C \vee \neg B \\ \hline A \vee C \end{array}$$

.

I called this "a direct consequence" because -perhaps somewhat arbitrarily- I would like to distinguish between inference rules (different applications of which may yield results that are not equivalent) and simplifications that are possible according to boolean algebra -such as replacing $B \wedge \neg B$ by <u>false</u> and $A \vee C \vee$ <u>false</u> by $A \vee C$ -, but never change the value of the boolean expression.

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The above caused me to revisit the problem of the nine mathematicians visiting an international congress, and about whom we are invited to prove

$$A \vee B \vee C \qquad (5)$$

with

A: there exists a triple of mathematicians that is incommunicado (i.e. such that no two of them have a language in common)

B: there exists a mathematician mastering more than three languages

C: there exists a language mastered by at least three mathematicians.

Very much like the introduction of (named!) auxiliary lines or points in geometry proofs, I propose to introduce named <u>auxiliary propositions</u>, such that we can prove lemmata connecting them to the above propositions, such as

D: there exists a mathematician that can communicate with more others than he masters languages,

for which we can prove

<u>Lemma 1</u> $C \vee \neg D$.

<u>Proof</u> Obvious. With this qualification we mean here that we can start as well with observing

$C \vee$ "each mathematician communicates in different languages with those others he can communicate with", etc.

as with observing

$\neg D \vee$ "there exists a mathematician that shares a language with at least two others", etc.

(End of proof of Lemma 1.)

With

E: there exists a mathematician that can communicate with more than three others,

we can prove

<u>Lemma 2</u>. $A \vee E$.

<u>Proof</u> Let "$x|y$" here stand for "x and y are two different mathematicians that have no language in common. With

G: for each x, the equation $x|u$ has at least five different solutions for u,

we observe (obviously)

$$E \vee G \qquad (6)$$

With

H: with y and z constrained to belong to an arbitrary quintuple, the equation $y|z$ has at least one solution in y and z,

we observe (equally obviously)

$$E \vee H \quad . \tag{7}$$

Applying rule (4) to assertions (6) and (7) we find $E \vee (G \wedge H)$,

hence

$E \vee$ "for each x, the equation $x|y \wedge x|z \wedge y|z$ has at least one solution in y and z".

(End of proof of Lemma 2)

Applying rule (4) to Lemmata 1 and 2 we infer the

<u>Corollary</u>. $A \vee C \vee (E \wedge \neg D)$.

Remembering rule (4) we see that (5) has been proved when we can prove $B \vee \neg(E \wedge \neg D)$ or, equivalently

<u>Lemma 3.</u> $B \vee D \vee \neg E$.

<u>Proof</u>. Obvious. (End of proof of Lemma 3).

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Note that in the above the Corollary was only used for heuristic purposes. Once Lemmata 1, 2, and 3 have been established we could have inferred

$$\frac{\begin{array}{c} A \vee E \\ B \vee D \vee \neg E \end{array}}{A \vee B \vee D} \quad \text{and} \quad \frac{\begin{array}{c} A \vee B \vee D \\ C \vee \neg D \end{array}}{A \vee B \vee C}$$

and our two individual inferences would have been of the traditional form of the transitive implication.

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I know that firm believers in the so-called "natural deduction" will state that, in the case of Lemma 2, I am just "deducing naturally" that A follows from the "assumption" ¬E. In this appreciation they will find themselves strengthened by the observation that in that proof all assertions start with "E v". They have a point, but the point is weak. Look at the structure of the proof as a whole. Lemmata 1, 2, and 3 capture it; from there rule (4) does the job, and at that level it is very arbitrary to subdivide assertions into assumptions and conclusions.

Remark. Observing the seven triples xyz for a pair (x, y) such that $x|y$, the argument proving Lemma 2 can equally well be phrased in terms of assertions starting with "A v". In the sense used above also Lemma 2 is obvious. (End of remark.)

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27 February 1980
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