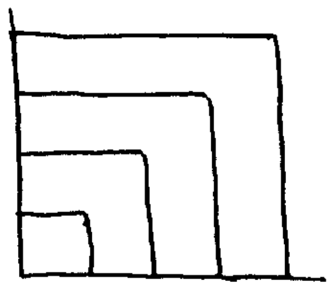
EWD857.html

Generalizing an old formula

It all started with a discussion about the role of pictures as substitute for calculational proofs. I was reminded of the pictorial illustration of the theorem that the sum of the first n odd natural numbers equals n^2:



Immediately the question rose whether this formula can be generalized to higher powers of n, since from the 4th power onwards, visualization breaks down.

By mentally cutting 3 mutually orthogonal slices of thickness 1 from a cube of n by n by n, one can convince oneself that

$$n^3 = (\underline{S}\, i : 1 \le i \le n : i^2 + i\cdot(i-1) + (i-1)^2)$$

and one begins to suspect the validity of

$$n^4 = (\underline{S}\, i : 1 \le i \le n : i^3 + i^2\cdot(i-1) + i\cdot(i-1)^2 + (i-1)^3)$$

and similar formula, in general

$$n^k = (\underline{S}\, i : 1 \le i \le n : (\underline{S}\, j : 0 \le j < k : i^{k-1-j}\cdot(i-1)^j)) . \qquad (0)$$

This holds provided

$$n^k = (\underline{S}\, j : 0 \le j < k : n^{k-1-j}\cdot(n-1)^j) + (n-1)^k \qquad (1)$$

because (0) holds for $n=0$, while (1) then provides the step for proving (0) by induction over n.

We prove (1), which holds for $k=1$, by mathematical induction over k:

$$n^k = (\underline{S}j: 0 \leq j < k: n^{k-1-j} \cdot (n-1)^j) + (n-1)^k$$

$\equiv$ {multiply both sides by n; $n = 1 + (n-1)$}

$$n^{k+1} = (\underline{S}j: 0 \leq j < k: n^{k-j} \cdot (n-1)^j) + (n-1)^k + (n-1)^{k+1}$$

$\equiv$ {definition of summation and $n^0 = 1$}

$$n^{k+1} = (\underline{S}j: 0 \leq j < k+1: n^{k+1-1-j} \cdot (n-1)^j) + (n-1)^{k+1}$$

q.e.d.

The above was just for the record. I find the above double induction satisfactory and formula (0) was new for me.

<u>Observation</u>. In the mean time I asked people from five different countries: none of them had seen the above picture at school. We all agreed that that was a pity. (End of Observation)

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