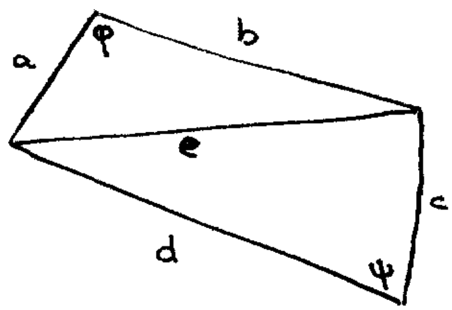
On the "quadrangular" inequalities

For any four line segments we have

(the segments can be the edges of a 4-gon) $\equiv$
(each of the segments is in length at most the sum of the lengths of the other three) .

Presumably, this is well-known; if not, it is rather obvious. I would never have bothered to write it down, were it not for the fact that I could use all four inequalities to demonstrate that, given the lengths of the edges of a 4-gon, the latter can be "bent" in such a way that its 4 vertices lie on a circle.



The condition that the vertices lie on a circle is in the above terminology equivalent to $\varphi + \psi = \pi$. We shall demonstrate the possibility by computing φ .

According to the cosine rule

$$e^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos.\varphi \qquad ;$$

similarly, and because $\cos.\psi = -\cos.\varphi$, we have

$$e^2 = c^2 + d^2 + 2 \cdot c \cdot d \cdot \cos.\varphi \qquad .$$

Eliminating e^2 , we find that $\cos.\varphi$ is a root of the equation

$$x: (a^2+b^2-(c^2+d^2) = (2\cdot a\cdot b + 2\cdot c\cdot d)\cdot x)$$

and, since $-1 \leq \cos.\varphi \leq +1$, we have to show that the -unique- solution to the above equation is in absolute value at most 1. Confining ourselves to the case that all four lengths are positive - if some lengths are zero, the theorem is trivial-, we have $2\cdot a\cdot b + 2\cdot c\cdot d > 0$, and our proof obligation boils down to demonstrating

(i) $a^2+b^2-(c^2+d^2) \leq 2\cdot a\cdot b + 2\cdot c\cdot d$ and

(ii) $a^2+b^2-(c^2+d^2) \geq -2\cdot a\cdot b - 2\cdot c\cdot d$.

$$
\begin{array}{ll}
 & \text{(i)} \\
= & \{\text{algebra}\} \\
 & a^2 - 2\cdot a\cdot b + b^2 \leq c^2 + 2\cdot c\cdot d + d^2 \\
= & \{\text{algebra}\} \\
 & (a\text{-}b \ \underline{\max}\ b\text{-}a)^2 \leq (c+d)^2 \\
\Leftarrow & \{\text{for non-negative base, the square is monotonic}\} \\
 & (a\text{-}b \ \underline{\max}\ b\text{-}a) \leq c+d \\
= & \{\text{arithmetic}\} \\
 & a-b \leq c+d \ \wedge\ b-a \leq c+d \\
= & \{\text{arithmetic}\} \\
 & a \leq b+c+d \ \wedge\ b \leq c+d+a \ .
\end{array}
$$

Similarly

$$
\begin{array}{ll}
 & \text{(ii)} \\
= & \{\text{algebra}\} \\
 & a^2 + 2\cdot a\cdot b + b^2 \geq c^2 - 2\cdot c\cdot d\cdot + d^2 \\
\Leftarrow & \{\text{the above with (a,b) and (c,d) interchanged}\} \\
 & c \leq d+a+b \ \wedge\ d \leq a+b+c \ .
\end{array}
$$

Hence the demonstrandum follows from all four quadrangular inequalities.

I think it a nice proof because I had never expected to use the quadrangular inequalities so explicitly. There is also something profoundly wrong with it because it cannot be readily generalized to polygons, for which the theorem is also valid. (It is, for given edge lengths, the polygon with maximum area; this has the property that its vertices lie on a circle.)

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