On W.H.J. Feijen's string equation

This note deals with a problem posed and solved by W.H.J. Feijen. It does not need to be written because one of these days he will write it down if he has not done so already. But I had an urge to write something technical and could not resist the challenge of trying to develop the little theory needed really nicely.

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We use capital letters to denote finite strings from some alphabet. In what follows, the specific choice of alphabet is so irrelevant that the latter can remain anonymous; in fact, the alphabet will not be mentioned any more.

Concatenation of strings will be denoted by juxtaposition; concatenation is associative and the empty string is its identity element.

The length function will be denoted by #.

We shall use the following lemmata (I think):

Lemma 0: The empty string is the only solution of the equation $X{:}(\#.X = 0)$.

Lemma 1: $\#.(XY) = \#.X + \#.Y$

Lemma 2: $\#.A = \#.C \wedge \#.B = \#.D \Rightarrow$
$(AB = CD \equiv A = C \wedge B = D)$

The task posed by Feijen is to investigate the equation

(0) $\quad X\colon\ (AX = XB)$,

i.e. to determine the conditions under which this equation is solvable and to determine under those conditions its solutions.

From Lemma 1 we immediately deduce

(1) $\quad AX = XB \ \Rightarrow\ \#.A = \#.B$,

hence our first result:

(2) $\quad \#.A \neq \#.B \ \Rightarrow$ (0) has no solutions .

From Lemma 0 and the fact that the empty string is the identity element of the concatenation, we derive our second result:

(3) $\quad \#.A = 0 \wedge \#.B = 0 \ \Rightarrow$ any string solves (0) .

Therefore, from now on we confine ourselves to the study of (0) with

(4) $\quad \#.A = n \ \wedge\ \#.B = n \ \wedge\ n > 0$.

I realize that I need a further lemma about concatenation:

<u>Lemma 3</u> With $n \leq \#.X$ the equation

$$Y, Z: (X = YZ \wedge \#.Z = n)$$

has a unique solution.

And now we are ready for the works. We observe under validity of (4) for any X, Y, and Z with $\#.Z = n$

$$
\begin{array}{ll}
 & (X \text{ solves } (0)) \wedge X = YZ \\
= & \{ (0) \text{ and predicate calculus/Leibniz} \} \\
 & AYZ = YZB \wedge X = YZ \\
= & \{ \text{Lemma 2} \} \\
 & AY = YZ \wedge Z = B \wedge X = YZ \\
= & \{ \text{predicate calculus/Leibniz} \} \\
 & AY = YB \wedge X = YB \wedge Z = B
\end{array}
$$

i.e. $X = YB$ defines a one-to-one correspondence between solutions differing n in length and we can now confine our study to solutions X of (0) with $\#X \leq n$. Under that constraint —lemma 3— we can write

(5) $\quad A = PQ \wedge B = RS \quad$ with $\#.P = \#.X \wedge \#S = \#.X$

and observe

$$
\begin{array}{ll}
 & (X \text{ solves } (0)) \\
= & \{ (0) \text{ and } (5) \} \\
 & PQX = XRS \\
= & \{ (5) \text{ and Lemma 2} \} \\
 & P = X \wedge Q = R \wedge X = S \\
= & \{ \text{calculus and } (5) \} \\
 & P = X \wedge A = PQ \wedge B = QP
\end{array}
$$

i.e. solvability of (0) is equivalent to solvability of (6) given by

(6) $\quad P, Q : (A = PQ \ \wedge \ B = QP)$

whose solvability expresses that A can be obtained by rotating B. Each solution P, Q of (6) generates the solutions PB^* of (0).

<u>Remark</u> A more symmetric way of expressing those solutions is A^*PB^*. (End of Remark.)

The problem is not deep, but it is nice it can be dealt with so nicely.

Nuenen, 25 Dec. 1986

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