My methodological blunder with grid polygons

A "grid polygon" is defined as a plane polygon whose vertices are the only grid points on its edges.

Lemma Consider a grid triangle with 1 interior grid point; the interior point is the centre of gravity of the triangle.

I thought for a while and constructed

 and  .

Asked about my progress, I showed these two, adding "but I hate the case analysis and have as yet not shown that it is exhaustive". How right was I about my caveat! The above case analysis is not exhaustive —there are infinitely many such triangles— and, moreover, it is totally superfluous.

I constructed these examples with the intention of convincing myself that grid triangles with 1 interior grid point do indeed exist. A stupid exercise, for if they don't exist, the theorem is vacuously correct; furthermore, it is notoriously ineffective to study a non-empty set by looking at some of its members.

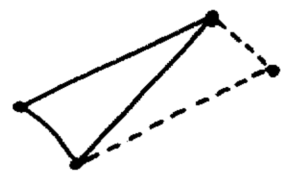
So we start afresh. The antecedent is about a grid triangle with 1 interior grid point, and the consequent is about a triangle and its centre of gravity. Start at the least familiar side, which is the antecedent. What can we say about a grid triangle with 1 interior point? By connecting the interior point with the vertices, we partition the original triangle into three grid triangles without interior grid points? A simpler concept!

What can we say about a grid triangle without interior grid points? In a way, it should not be too big, but if we are at a loss as how to make this constraint more precise, we can look at the consequent: what can we say about the size of the three triangles a big trangle is partitioned into when the centre of gravity is connected to the three vertices? The three small triangles have the same area, and, conversely, when a triangle is thus partitioned into three triangles of equal area, the interior point is the centre of gravity.

Since a grid triangle cannot be too small either we make a leap of faith and try to prove our theorem by showing that *all* grid triangles without interior grid points have the *same* area. Looking at the simplest example

we conclude that that area would have to be ½.

This fractional value begs to be eliminated, for, after all, grid points are about integer values. Fortunately, each grid triangle is half a grid parallelogram



and if the grid triangle is without interior grid point, so is the grid parallelogram. So we are left with the obligation to show that a grid parallelogram without interior grid points has an area 1. This is most briefly shown by tiling the plane by such parallelograms, thus establishing a one-to-one correspondence between parallelograms and grid points — say, each parallelogram and its top-right vertex.

I am endebted to Wolfgang Hinderer (from Karlsruhe) for drawing my attention to this argument.

Nuenen, 23 August 1988

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