It is all distributivity

Let $\bullet$ be a binary infix operator. Depending on how we Curry, we can form two classes of unary operators from, the general elements of which are often denoted by $(x\bullet)$ and $(\bullet y)$, defined

$$(x\bullet) = \langle\lambda y: x\bullet y\rangle \qquad (\bullet y) = \langle\lambda x: x\bullet y\rangle \quad .$$

We call them in this note "the derived unary operators".

To begin with we observe — with apologies for the two different usages of parentheses —

$$\begin{array}{ll} & (x\bullet z)\bullet y \;=\; x\bullet(z\bullet y) \\ = & \quad \{\text{definition of derived operators}\} \\ & (\bullet y).((x\bullet).z) \;=\; (x\bullet).((\bullet y).z) \\ = & \quad \{\text{definition of functional composition}\} \\ & ((\bullet y)\circ(x\bullet)).z \;=\; ((x\bullet)\circ(\bullet y)).z \qquad , \end{array}$$

hence, that a binary operator is "associative" means that its derived unary operators commute.

To say that $\bullet$ "distributes over" some binary operator is really a statement about its derived unary operators: "$\bullet$ distributes from the left over $\square$" means that for all x, y, z:

$$(x\bullet).(y \,\square\, z) \;=\; (x\bullet).y \;\square\; (x\bullet).z \;;$$

distribution from the right means similarly

$$(\bullet x).(y \,\square\, z) \;=\; (\bullet x).y \;\square\; (\bullet x).z \quad .$$

Both formulae are of the form

$$f.(y \,\square\, z) \;=\; f.y \;\square\; f.z \quad ,$$

which captures "f distributes over $\square$". But what comes of this formula if $\square$ turns out to be unary? Replacing in the last formula $p \,\square\, q$ by $g.p$, we get

$$f.(g.y) = g.(f.y) \qquad \text{or} \qquad f\circ g = g\circ f \quad .$$

Hence, that two unary operators "commute" means that they distribute over each other. Hence, that a binary operator is associative just means that its derived unary operators distribute over each other.

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