Junctivity and massaging quantification

To begin with, we discuss universal disjunctivity. That predicate transformer f is universally disjunctive means that for any range of the dummy and any p (of the appropriate type)

(0) $\quad [f.\langle \exists x :: p.x \rangle \equiv \langle \exists x :: f.(p.x) \rangle]$.

The question we raise is whether universal disjunctivity of f follows from the fact that for any range of the dummy y

(1) $\quad [f.\langle \exists y :: y \rangle \equiv \langle \exists y :: f.y \rangle]$,

i.e. whether in order to demonstrate (0), it suffices to confine one's attention to the specific instantiation $p :=$ "identity function".

The answer to this question is positive on account of the following theorem:

Theorem 0. For any (appropriately typed) p, q, r

(2) $\quad [\langle \exists x : r.x : q.(p.x) \rangle \equiv \langle \exists y : C.r.p.y : q.y \rangle]$

(3) $\quad [\langle \forall x : r.x : q.(p.x) \rangle \equiv \langle \forall y : C.r.p.y : q.y \rangle]$

where C is given by

(4) $\quad [C.r.p.y \equiv \langle \exists x : r.x : [p.x = y] \rangle]$

<u>Proof</u> In order to prove (2), we observe for any p, q, r - in great detail -

$$
\begin{array}{ll}
 & \langle \exists y : C.r.p.y : q.y \rangle \\
= & \quad \{ (4) \} \\
 & \langle \exists y : \langle \exists x : r.x : [p.x = y] \rangle : q.y \rangle \\
= & \quad \{ \text{trading} \} \\
 & \langle \exists y :: \langle \exists x : r.x : [p.x = y] \rangle \wedge q.y \rangle \\
= & \quad \{ \wedge \text{ over } \exists \} \\
 & \langle \exists y :: \langle \exists x : r.x : [p.x = y] \wedge q.y \rangle\rangle \\
= & \quad \{ \text{trading} \} \\
 & \langle \exists y :: \langle \exists x : r.x \wedge [p.x = y] : q.y \rangle\rangle \\
= & \quad \{ \text{interchange of quantifications} \} \\
 & \langle \exists x : r.x : \langle \exists y : [p.x = y] : q.y \rangle\rangle \\
= & \quad \{ \text{1-point rule} \} \\
 & \langle \exists x : r.x : q.(p.x) \rangle \quad .
\end{array}
$$

The crucial observation is that the new range $C.r.p$ does <u>not</u> depend on q. This allows us to prove (3) by instantiating (2) with $q := \neg q$, and then applying de Morgan's Law.
(End of Proof.)

And now the ground work has been done to derive (0) from (1), more precisely: we observe for an f satisfying (1) for any range of the dummy y, and any r, p of the appropriate types

$$
\begin{array}{ll}
& f.\langle \exists x : r.x : p.x \rangle \\
= & \quad \{ (2) \text{ with } q := id \} \\
& f.\langle \exists y : C.r.p.y : y \rangle \\
= & \quad \{ (1) \text{ with range: } C.r.p \} \\
& \langle \exists y : C.r.p.y : f.y \rangle \\
= & \quad \{ (2) \text{ with } q := f \} \\
& \langle \exists x : r.x : f.(p.x) \rangle \qquad .
\end{array}
$$

I owe Theorem 0 to the ETAC, which considers it (I think) as a generalization of "splitting the range". My interest is here in the relation between (0) and (1). It is of the form

(5) $\qquad \langle \forall p :: \mathcal{B}.p \rangle \equiv \mathcal{B}.id \qquad ;$

to demonstrate $\mathcal{B}$, one demonstrates the RHS; to use $\mathcal{B}$, one uses the LHS which can be instantiated as you like.

Situation (5) is common, and for methodological reasons we should know and recognize it. For instance

$$\langle \forall x :: [c;x \Rightarrow x] \rangle \equiv [c \Rightarrow J]$$

gives us in the relational calculus two ways of expressing that c is "a midde condition" (or "a monotype"). Similarly

$$\langle \forall x :: [x;r \Rightarrow r] \rangle \equiv [true; r \Rightarrow r]$$

gives us two ways of expressing that r

is "a right condition". Finally

$$\langle \forall i,j : i \le j : A.i \le A.j \rangle \quad \equiv \quad \langle \forall i :: A.i \le A.(i+1) \rangle$$

gives us two ways of expressing that the sequence A is ascending.

The last three examples rely for LHS $\Leftarrow$ RHS on transitivity or monotonicity. This note has been written because the first example, i.e. the characterizations of junctivity, seems not to do so.

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