Another forced ping-pong argument?

I owe the following theorem to Rutger (see [0]): for any relation f

(0) $[f;\neg J \equiv \neg f] \equiv \langle\forall y::[f;\neg y \equiv \neg(f;y)]\rangle$.

I think this quite a remarkable theorem. The right-hand side expresses that the prefix operators "$f;$" and "$\neg$" commute, i.e. that the functions $(f;)\circ(\neg)$ and $(\neg)\circ(f;)$ are the same; the left-hand side only expresses that these two functions yield the same value when applied to J.

Now the crucial observation is that at both sides an expression monotonic in f equivales an expression antimonotonic in f. Mutual implication turns such equivalences into monotonic and antimonotonic conjuncts; (0) can be rewritten as

(1) $[\neg f \Rightarrow f;\neg J] \wedge [f;\neg J \Rightarrow \neg f] \quad \equiv$

$\qquad \langle\forall y::[\neg(f;y) \Rightarrow f;\neg y]\rangle \wedge \langle\forall y::[f;\neg y \Rightarrow \neg(f;y)]\rangle$

and now it stands to reason to try to equate the two monotonic conjuncts and to equate the two antimonotonic conjuncts, i.e. to prove separately

(2) $[\neg f \Rightarrow f;\neg J] \equiv \langle\forall y :: [\neg(f;y) \Rightarrow f;\neg y]\rangle$

(3) $[f;\neg J \Rightarrow \neg f] \equiv \langle\forall y :: [f;\neg y \Rightarrow \neg(f;y)]\rangle$

This, indeed, can be done and for the sake of completeness we include the two proofs

<u>Proofs</u> We observe for any f

$$\begin{array}{ll}
 & \langle\forall y :: [\neg(f;y) \Rightarrow f;\neg y]\rangle \\
= & \quad \{\text{pred. calc.}\} \\
 & \langle\forall y :: [f;y \ \vee\ f;\neg y]\rangle \\
= & \quad \{\text{rel. calc.}\} \\
 & \langle\forall y :: [f;(y \vee \neg y)]\rangle \\
= & \quad \{\text{pred. calc.}\} \\
 & \langle\forall y :: [f;(J \vee \neg J)]\rangle \\
= & \quad \{\text{pred. calc., range of } y \text{ non-empty}\} \\
 & [f;(J \vee \neg J)] \\
= & \quad \{\text{rel. calc.}\} \\
 & [f \ \vee\ f;\neg J] \\
= & \quad \{\text{pred. calc.}\} \\
 & [\neg f \Rightarrow f;\neg J]
\end{array}$$

and

$$\begin{array}{ll}
 & \langle\forall y :: [f;\neg y \Rightarrow \neg(f;y)]\rangle \\
= & \quad \{\text{rel. calc.}\} \\
 & \langle\forall y :: [\sim f;f;y \Rightarrow y]\rangle \\
= & \quad \{\Rightarrow \text{ inst. } y:=J\}\ \{\Leftarrow \text{ monotonicity of } ;\} \\
 & [\sim f;f \Rightarrow J] \\
= & \quad \{\text{rel. calc.}\} \\
 & [f;\neg J \Rightarrow \neg f]
\end{array}$$

(End of Proofs.)

The reason for writing this note is to point out that the transition from (1) to (2) $\wedge$ (3), though "it stands to reason" and usually seems sound heuristics, is less forced than I had expected (and hoped) to be. Look at the following examples.

Suppose that for some p and q we have to prove for any real x

(4) $\quad 3 \leq x \;\wedge\; x \leq 10 \;\equiv\; p \leq x \;\wedge\; x \leq q$.

Viewing $\langle \forall x :: (4) \rangle$ as an equation in p, q, $(p,q) = (3,10)$ is its only solution, and, hence, we can replace the proof obligation (4) by the obligation to show that for any x

(5) $\quad 3 \leq x \;\equiv\; p \leq x$
(6) $\quad x \leq 10 \;\equiv\; x \leq 10$,

i.e. we equate the monotonic conjuncts and, separately, the antimonotonic ones.

But suppose now that, instead of (4) we had to prove

(7) $\quad x \leq 3 \;\wedge\; 10 \leq x \;\equiv\; x \leq p \;\wedge\; q \leq x$.

Replacing this analogously by the obligation to prove for all x

$$(8) \qquad x \leq 3 \;\equiv\; x \leq p$$
$$(9) \qquad 10 \leq x \;\equiv\; q \leq x$$

gives a much stronger obligation, since

$$\langle \forall x :: (7) \rangle \;\equiv\; p < q \quad .$$

Currently I don't see how to express that (1) is *more* like (4) than like (7).

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