Generators of extreme values

"s generates the strongest value of p" means

(0) $\langle \forall x :: [p.s \Rightarrow p.x] \rangle$;

"w generates the weakest value of p" means

(1) $\langle \forall x :: [p.x \Rightarrow p.w] \rangle$.

For arbitrary p , these extreme-value generators need not exist, and if they exist, they need not be unique. Notice that p need not have any monotonicity property. From (0) and (1) follow (by predicate calculus)

$[p.s \Rightarrow \langle \forall x :: p.x \rangle]$ and $[\langle \exists x :: p.x \rangle \Rightarrow p.w]$

respectively, which imply by instantiation

(2) $[p.s \equiv \langle \forall x :: p.x \rangle]$

(3) $[\langle \exists x :: p.x \rangle \equiv p.w]$.

These formulae are of potential interest because they equate a quantified expression to a non-quantified one.

Remark Note (0) $\equiv$ (2) and (1) $\equiv$ (3). (End of Remark.)

In the case of monotonic m , (0) implies

(4) $\langle \forall x :: [m.(p.s) \Rightarrow m.(p.x)] \rangle$

i.e. s generates the strongest value of $m \circ p$. In the case of anti-monotonic am, however, (1) implies

(5) $\langle \forall x :: [am.(p.w) \Rightarrow am.(p.x)] \rangle$,

i.e w generates the strongest value of $am \circ p$.

<u>Example 0</u> In the case

$$[p.x \equiv \text{constant}] \qquad (\text{for all } x),$$

both s and w exist and arbitrary values can be chosen for them. For instance, in EWD 1195 we saw an example with

$$[p.x \equiv x \vee \neg x] ,$$
$$[s \equiv J]$$
$$m.y \equiv [f; y]$$

J generates the strongest value of $m \circ p$, that is (2) with $p := m \circ p$. Since

$$(m \circ p).x \equiv [\neg(f;x) \Rightarrow f; \neg x]$$

– see EWD 1195 or prove it yourself – this yields

$$[\neg(f;J) \Rightarrow f;\neg J] \equiv \langle \forall x :: [\neg(f;x) \Rightarrow f;\neg x] \rangle$$

which is EWD 1195 (2).

Example 1 In the case of the identity function $[p.x \equiv x]$ (for all x)

both s and w exist and are unique:

$$[s \equiv false] \text{ and } [w \equiv true] .$$

For example, (see EWD1191), with

$$[p.x \equiv x]$$

$$[w \equiv true]$$

$$am.y \equiv [y;r \Rightarrow r] ,$$

w generates the strongest value of $am \circ p$, i.e. (2) with $p, s := am \circ p, true$:

$$[true; r \Rightarrow r] \equiv \langle \forall x :: [x; r \Rightarrow r] \rangle ,$$

i.e. the two ways of expressing that r is a "right condition".

Example 2 We begin by observing

$$true$$
$$= \quad \{ J \text{ neutral element of composition} \}$$
$$\langle \forall x :: [J; x \Rightarrow x] \rangle$$
$$= \quad \{ \text{left exchange} \}$$
$$\langle \forall x :: [\neg x ; \sim x \Rightarrow \neg J] \rangle$$
$$= \quad \{ \sim J \text{ also neutral element of composition} \}$$

(6) $\langle \forall x :: [\neg x ; \sim x \Rightarrow \neg J ; \sim J] \rangle$

that is:

with
$$[p.x \equiv \neg x ; \sim x] \quad (\text{for all } x)$$
we have
$$[w \equiv J] \quad .$$

Now, with
$$am.y \equiv [y \Rightarrow \neg c]$$
J generates the strongest value of $am \circ p$.
Since
$$(am \circ p).x \equiv [c ; x \Rightarrow x]$$
we conclude
$$[c ; J \Rightarrow J] \equiv \langle \forall x :: [c ; x \Rightarrow x] \rangle \quad ,$$
i.e. the equivalence of two ways of expressing that c is a "middle condition".

<u>Remark</u> With the specific choice
$$[c \equiv \sim f ; f]$$
we get
$$(am \circ p).x \equiv [f ; \neg x \Rightarrow \neg(f ; x)]$$
and our last conclusion becomes
$$[f ; \neg J \Rightarrow \neg(f ; J)] \equiv \langle \forall x :: [f ; \neg x \Rightarrow \neg(f ; x)] \rangle \, ,$$
which is EWD1195 (3). (End of Remark.)

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prof. dr Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188, USA