# Triggered by Wim Feijen's treatment of "$\exists\forall \Rightarrow \forall\exists$"

In WF189, WHJ Feijen presents totally convincing heuristics for the derivation of Carroll Morgan's proofs of the well-known theorem from the predicate calculus

(0) $[\langle \exists x :: \langle \forall y :: t.x.y \rangle\rangle \Rightarrow \langle \forall y :: \langle \exists x :: t.x.y \rangle\rangle]$ .

These proofs use the Laws of the Quantified Constant – Wim Feijen regretted their anonymity, here is my suggestion for a name –

(1a) $[Q \Rightarrow \langle \forall z :: Q \rangle]$ for fresh $z$

(1b) $[Q \Leftarrow \langle \exists z :: Q \rangle]$ and any range ,

and the Laws of Instantiation, which are so well known that I won't write them down (now).

Reduced to its bare essentials, Carroll Morgan's proof is as follows

$$
\begin{array}{ll}
 & \langle \exists x :: \langle \forall y :: t.x.y \rangle\rangle \\
\Rightarrow & \quad \{ \text{Quantified Constant} \} \\
 & \langle \forall y' :: \langle \exists x :: \langle \forall y :: t.x.y \rangle\rangle\rangle \\
\Rightarrow & \quad \{ \text{Instantiation } y := y' \} \\
 & \langle \forall y' :: \langle \exists x :: t.x.y' \rangle\rangle \quad .
\end{array}
$$

But, perhaps, this proof has been reduced a little bit too much, for we really did as if the unmentioned ranges were true , but

in its full glory, the theorem to be proved is

(2) $$[\ \langle \exists x: R.x: \langle \forall y: S.y: t.x.y \rangle \rangle \Rightarrow \langle \forall y: S.y: \langle \exists x: R.x: t.x.y \rangle \rangle\ ]\ ,$$

and the Laws of Instantiation which I now give, are, when we include the ranges, (for instance)

(3a) $$[\ r.y \wedge \langle \forall z: r.z: f.z \rangle \Rightarrow f.y\ ]$$

(3b) $$[\ f.y \Rightarrow (r.y \Rightarrow \langle \exists z: r.z: f.z \rangle)\ ]\ .$$

For Carroll Morgan's proof, with ranges included, I suggest the following text:

$\langle \exists x: R.x: \langle \forall y: S.y: t.x.y \rangle \rangle$

$\Rightarrow$ { Quantified Constant }

$\langle \forall y': S.y': \langle \exists x: R.x: \langle \forall y: S.y: t.x.y \rangle \rangle \rangle$

$=$ { Range Diffusion, see below }

$\langle \forall y': S.y': \langle \exists x: R.x: S.y' \wedge \langle \forall y: S.y: t.x.y \rangle \rangle \rangle$

$\Rightarrow$ { Instantiation $y := y'$, according to (3a) }

$\langle \forall y': S.y': \langle \exists x: R.x: t.x.y' \rangle \rangle$

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We observe for any punctual $f$ and predicates $Q, X, Y$

$[\ Q \wedge f.X \equiv Q \wedge f.Y\ ]$ •

$=$ { pred. calc. }

$[\ Q \Rightarrow f.X \equiv Q \Rightarrow f.Y\ ]$ •

$=$ { pred. calc }

$[\ Q \Rightarrow (f.X \equiv f.Y)\ ]$

$\Leftarrow$ $\quad$ {$f$ is punctual}
$\quad [Q \Rightarrow (X \equiv Y)]$
$\Leftarrow$ $\quad$ {pred. calc}
$\quad [Y \equiv Q \wedge X] \;\vee\; [Y \equiv Q \Rightarrow X]$ •

Eliminating $Y$ from the lines marked with a bullet, we conclude for punctual $f$

(4a) $\quad [Q \wedge f.X \equiv Q \wedge f.(Q \wedge X)]$

(4b) $\quad [Q \wedge f.X \equiv Q \wedge f.(Q \Rightarrow X)]$

(4c) $\quad [Q \Rightarrow f.X \equiv Q \Rightarrow f.(Q \wedge X)]$

(4d) $\quad [Q \Rightarrow f.X \equiv Q \Rightarrow f.(Q \Rightarrow X)]$ .

Note that the prefix $Q\wedge$ creates a predicate that depends monotonically on $Q$, whereas $Q\Rightarrow$ creates antimonotonic dependence.

A consequence of (4) -and trading- is that the value of a universal or an existential quantification remains unchanged if we select a subexpression on which the term depends punctually and strengthen that subexpression by the prefix "range $\wedge$" or weaken it by the prefix "range $\Rightarrow$". This property is referred to by "Range diffusion".

In its applications it is, of course, useful to know that predicates built from $\neg, \vee, \wedge, \Rightarrow, \Leftarrow, \equiv, \not\equiv, \forall, \exists$ depend punctually on their

subexpressions (but since this has to be proved by induction over the syntax, I am afraid that Gries and Schneider would call this a metatheorem).

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When we have integer dummies $i,j$ with associated ranges $0 \leq i$ and $0 \leq j$, we usually dont manipulate those ranges: we replace the ranges by (an invisible) true and declare the dummies to be of <u>type</u> natural number and allow this type information to diffuse all through their scope. That's how we do it.

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA