A few addenda to EWD 1192

(This text is not self-contained. You should have EWD 1192: "The very first beginnings of lattice theory" at hand.)

Before "twinkle, twinkle, little star" on page EWD 1192 - 2, I should have drawn one further conclusion from axioms (0) through (3) and definition (5), viz.

(5a) $\quad z \leq x{\downarrow}y \;\equiv\; z \leq x \;\wedge\; z \leq y$

and its dual

(5b) $\quad x{\uparrow}y \leq z \;\equiv\; x \leq z \;\wedge\; y \leq z \quad .$

Proof of (5a) We observe for any x, y, z

$$\begin{array}{ll}
 & z \leq x \;\wedge\; z \leq y \\
= & \quad \{ (5), \leq \text{ in terms of } \downarrow \} \\
 & x{\downarrow}z = z \;\wedge\; y{\downarrow}z = z \\
= & \quad \{ \text{Leibniz} \} \\
 & x{\downarrow}y{\downarrow}z = z \;\wedge\; y{\downarrow}z = z \\
= & \quad \{ \text{Leibniz} \} \\
 & x{\downarrow}y{\downarrow}z = z \;\wedge\; y{\downarrow}x{\downarrow}y{\downarrow}z = z \\
= & \quad \{ (0), (1), \text{ and } (2) \} \\
 & x{\downarrow}y{\downarrow}z = z \;\wedge\; x{\downarrow}y{\downarrow}z = z \\
= & \quad \{ \text{idempotence of } \wedge, (5) \} \\
 & z \leq x{\downarrow}y
\end{array}$$

(End of Proof of 5(a))

The proof of (5b) is mutatis mutandis the same (replace $\downarrow$ by $\uparrow$ and $\leq$ by $\geq$).

The importance of (5a) is that it proves the solvability of equation (7a); similarly for (5b) and equation (7b). And this repairs my omission of showing that axiom (7) of the second system is a theorem of the first system.

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Jan L.A. van de Snepscheut drew my attention to the fact that, in the first system, we don't need to postulate (0), the idempotence of $\uparrow$ and $\downarrow$, because it follows from the laws of absorption. (Depending on how the laws of absorption are defined, one might need the symmetry. Associativity is <u>not</u> involved.)

We observe for any p, q

$p \uparrow p$

$=$ { left L. of A. with $x, y := p, q$ }

$p \uparrow (p \downarrow (q \uparrow p))$

$=$ { right L. of A. with $x, y := (q \uparrow p), p$ and symmetry of $\downarrow$ }

p

By interchanging or not the arguments of $\uparrow$ and/or of $\downarrow$, each law of absorption and theorem (4) can be written in 4 ways. For each formulation of (4) we can formulate the laws of absorption in such a way that (4) can be proved without appeal to (1), the symmetry of $\uparrow$ and $\downarrow$, but none of those formulations allows us to prove the idempotence of $\uparrow$ and $\downarrow$ without an appeal to (1). As a matter of fact, we then need the symmetry of $\uparrow$ to demonstrate the idempotence of $\downarrow$, and vice versa. (Please don't ask me how I established these last results. It was rather ugly, but polishing up the argument is insufficiently rewarding. Thank you for not asking.)

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