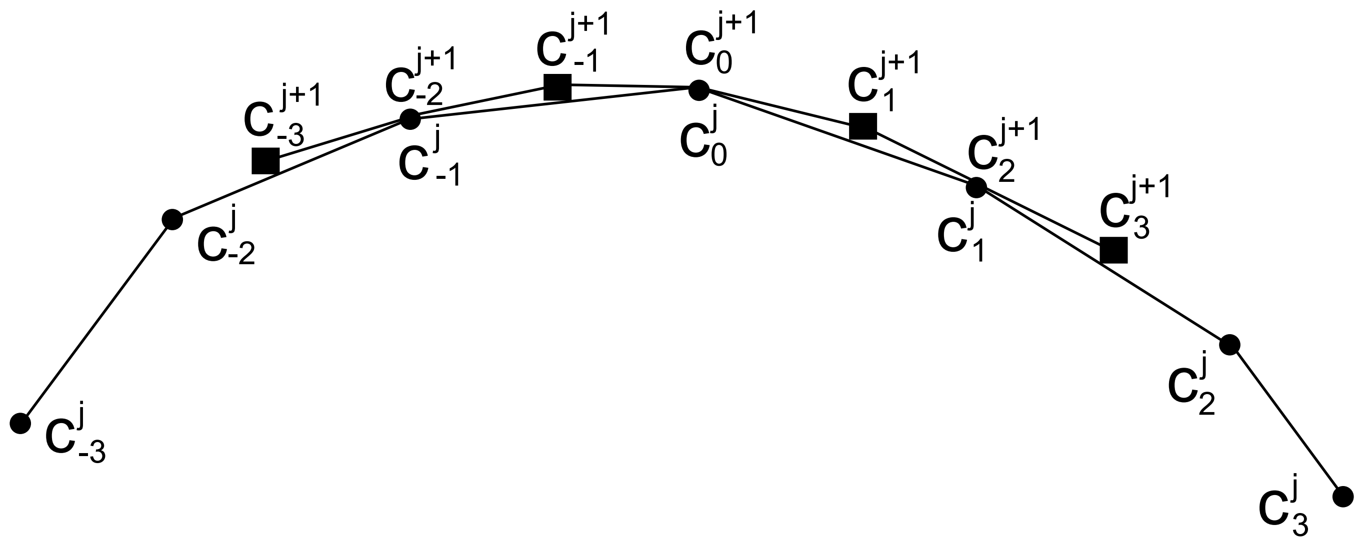
Recursive Subdivision and Iterated Functions

Subdivision of Curves

Four Point Scheme



Four point scheme: the filled circles are the level j control points, the filled squares are the level $j+1$ control points.

For four-point scheme we need to consider only 7 control points; these 7 points completely define the piece of the curve around a control point. We can consider a set of 7 control points *on any subdivision level*, as we do not care how small our piece of the curve is. Note that we can compute the positions of the seven control points on level $j+1$ from the positions of similar seven control points on level j, using a 7×7 submatrix $\mathbf{S}$ of the infinite subdivision matrix.

The local subdivision matrix for the four-point scheme is:

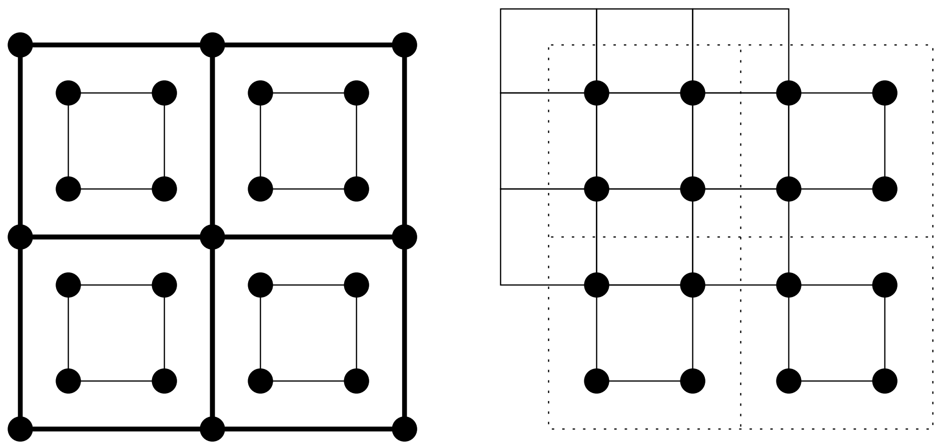
$$\begin{pmatrix} c_{-3}^{j+1} \\ c_{-2}^{j+1} \\ c_{-1}^{j+1} \\ c_{0}^{j+1} \\ c_{1}^{j+1} \\ c_{2}^{j+1} \\ c_{3}^{j+1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{pmatrix} \begin{pmatrix} c_{-3}^{j} \\ c_{-2}^{j} \\ c_{-1}^{j} \\ c_{0}^{j} \\ c_{1}^{j} \\ c_{2}^{j} \\ c_{3}^{j} \end{pmatrix}$$

Subdivision for Surfaces

Doo-Sabin

Split Rule for Quadrilaterals

New vertices are added to create level $j+1$ quadrilaterals in the center of level j quadrilaterals. The vertices of the center face quadrilaterals are connected to their neighbors. Each level j quadrilateral is covered by parts of 9 level $j+1$ quadrilaterals, but there are only 4 level $j+1$ quadrilaterals created for each level j quadrilateral. The vertices of the level j quadrilaterals are discarded. This refinement rule works for even degree tensor-product splines.



Refinement rule used by Doo-Sabin subdivision scheme.

$$P_0^{j+1} = \sum_{i=0}^{3} \alpha_i P_0^j$$

where

$$\alpha_0 = \frac{9}{16}, \quad \alpha_1 = \frac{3}{16}, \quad \alpha_2 = \frac{1}{16}, \quad \alpha_3 = \frac{3}{16}.$$

Subdivision Matrix

The matrix representation of Doo-Sabin subdivision scheme is (locally):

$$\begin{pmatrix} P_0^{j+1} \\ P_1^{j+1} \\ P_2^{j+1} \\ P_3^{j+1} \end{pmatrix} = \begin{pmatrix} \frac{9}{16} & \frac{3}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} & \frac{9}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{1}{16} & \frac{3}{16} & \frac{9}{16} \end{pmatrix} \begin{pmatrix} P_0^{j} \\ P_1^{j} \\ P_2^{j} \\ P_3^{j} \end{pmatrix}$$

N-gons

For each N-gon at level j, we create a level $j+1$ N-gon. Suppose we are computing a new vertex of the N-gon on level $j+1$. This vertex is a linear combination of the vertices of the old N-gon. Suppose these vertices are numbed from 0 to $N-1$ starting with the vertex nearest to the vertex on level $j+1$ that we are computing.

$$P_0^{j+1} = \sum_{i=0}^{N} \alpha_i P_0^j$$

where k is the vertex number and

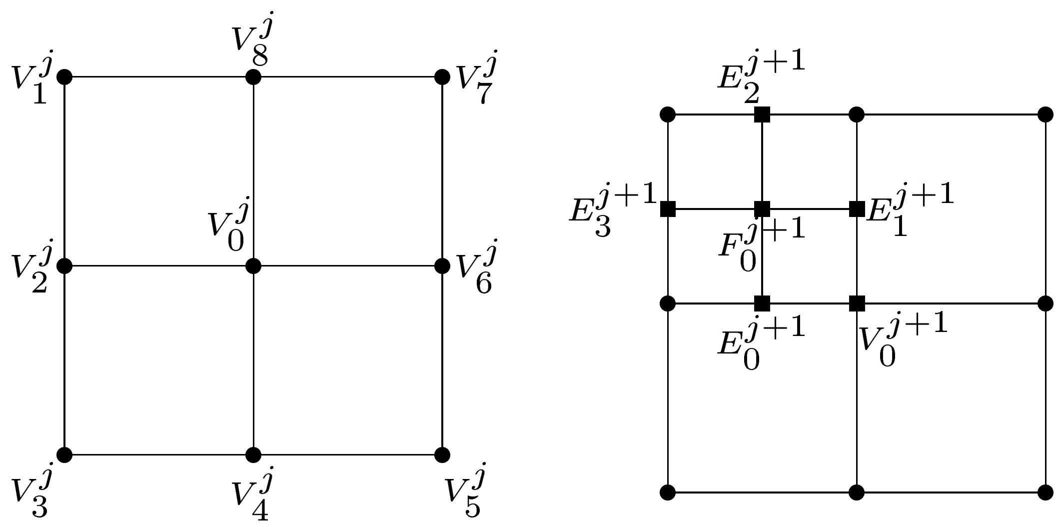
$$\alpha_0 = \frac{1}{4} + \frac{5}{4N},$$
$$\alpha_k = \frac{3 + 2\cos(\frac{2\pi k}{N})}{4N}, k = 1, \cdots, N-1$$

The matrix form in general is

$$\begin{pmatrix} P_0^{j+1} \\ P_1^{j+1} \\ \vdots \\ P_{N-1}^{j+1} \\ P_N^{j+1} \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{N-1} & \alpha_N \\ \alpha_N & \alpha_0 & \cdots & \alpha_{N-2} & \alpha_{N-1} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_N & \alpha_{N-1} & \cdots & \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N & \alpha_0 \end{pmatrix} \begin{pmatrix} P_0^j \\ P_1^j \\ \vdots \\ P_{N-1}^j \\ P_N^j \end{pmatrix}$$

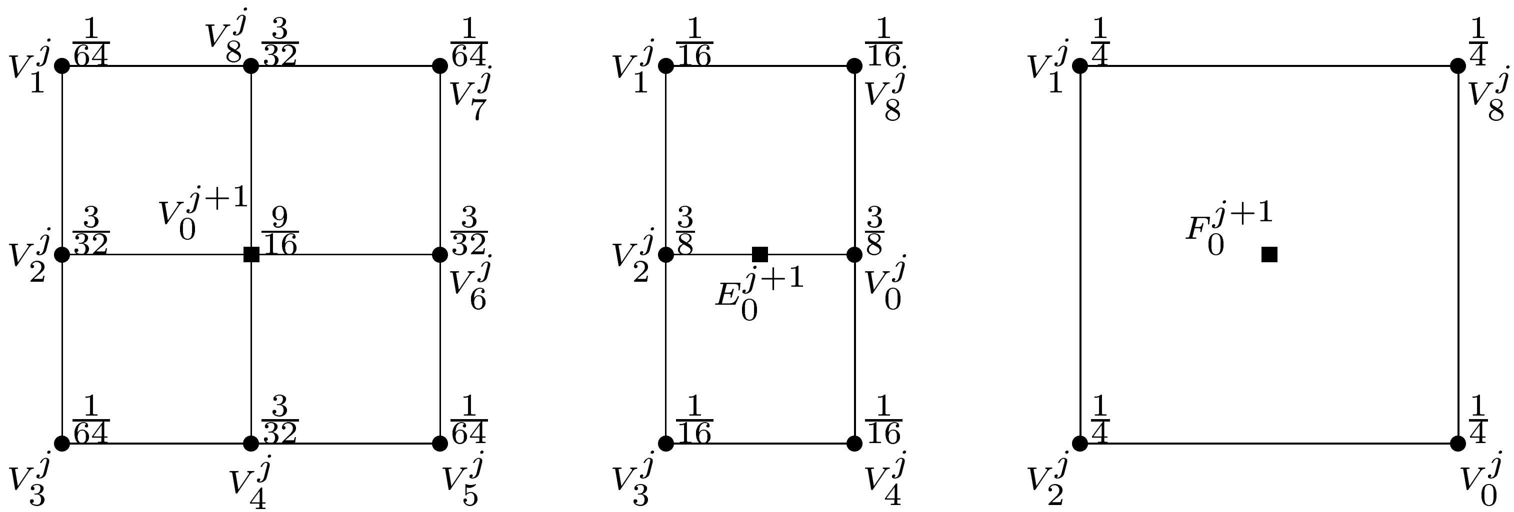
Catmull Clark

Refinement rule used by Catmull-Clark subdivision scheme is as follows. New vertices are added on each edge and in the center. When connected, 4 new level $j+1$ quadrilaterals are produced from the single level j quadrilateral.



Catmull-Clark subdivision scheme. Circles are the j level and Squares are the $j+1$ level.

The vertex rule, edge rule and face rule are shown in the following figure. Each black circle represents a vertex at level j; we compute the position of the vertex at level $j+1$ marked by the black square. Note that for the vertex rule, the control vertex with weight $\frac{9}{16}$ and the new vertex aren't necessarily aligned as they are in the figure.



- Vertex rule:

$$V_0^{j+1} = \frac{9}{16}V_0^j + \frac{3}{32}(V_2^j + V_4^j + V_6^j + V_8^j) + \frac{1}{64}(V_1^j + V_3^j + V_5^j + V_7^j)$$

- Edge rule:

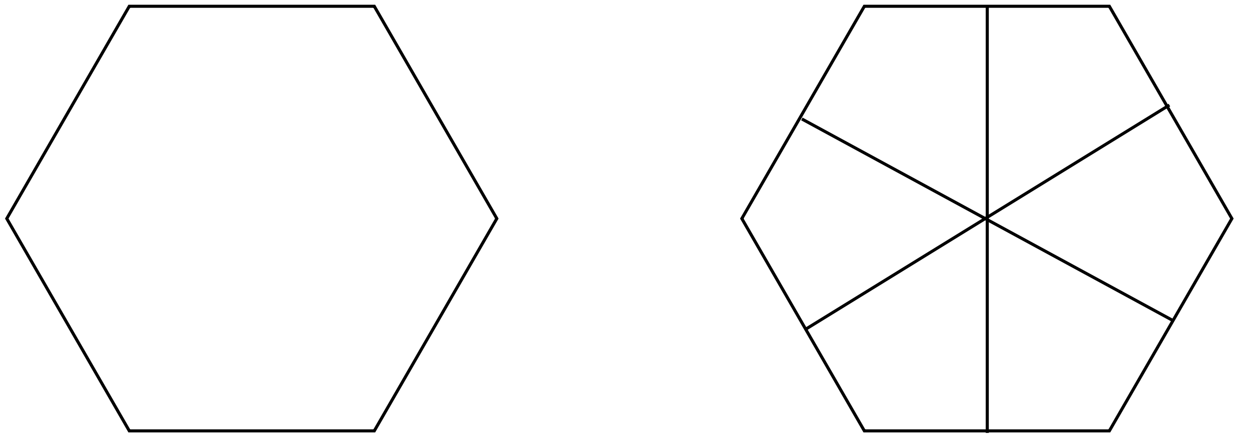
$$E_1^{j+1} = \frac{3}{8}(V_0^j + V_2^j) + \frac{1}{16}(V_1^j + V_3^j + V_4^j + V_8^j)$$

- Face rule:

$$F_0^{j+1} = \frac{1}{4}(V_1^j + V_2^j + V_0^j + V_8^j)$$

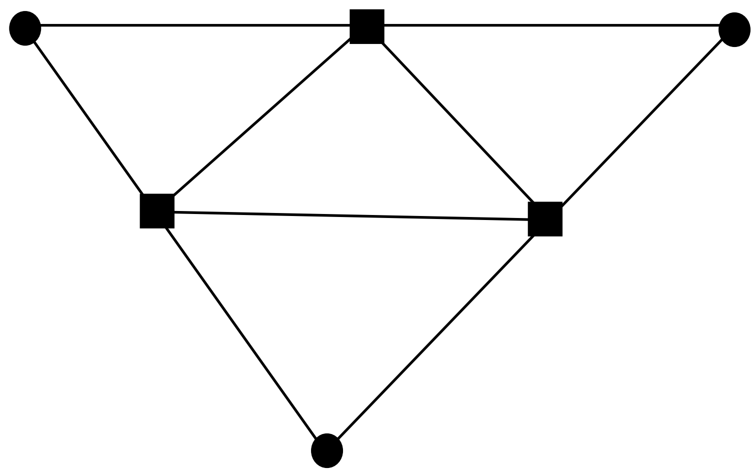
Arbitrary Meshes

We have defined Catmull-Clark scheme on quadrilaterals; it can be extended to handle arbitrary polygonal meshes. Observe that if we do one step of refinement, splitting each edge into two and inserting a new vertex for each face (see below Figure), we get a mesh which has only quadrilateral faces. On all other steps of subdivision standard rule described above can be applied.



Splitting a hexagon into quadrilaterals.

Loop



Here the filled circles indicate the old vertices, the filled squares indicate the new vertices.

Split Rule (Ordinary Vertex)

Loop subdivision rules for vertex (left) and edge (right) points at a regular vertex of degree 6.

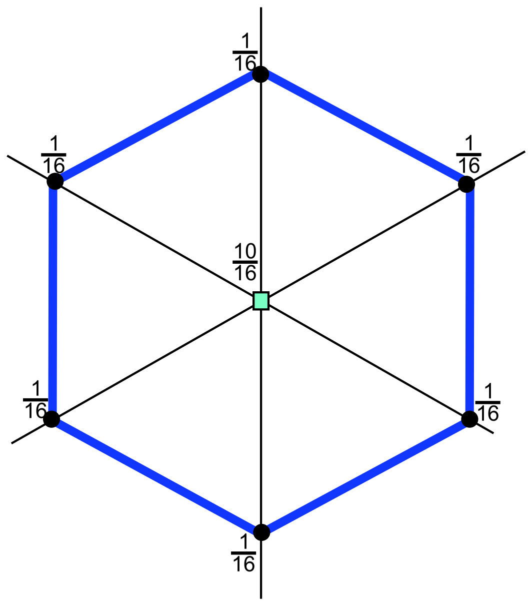
- Vertex rule:

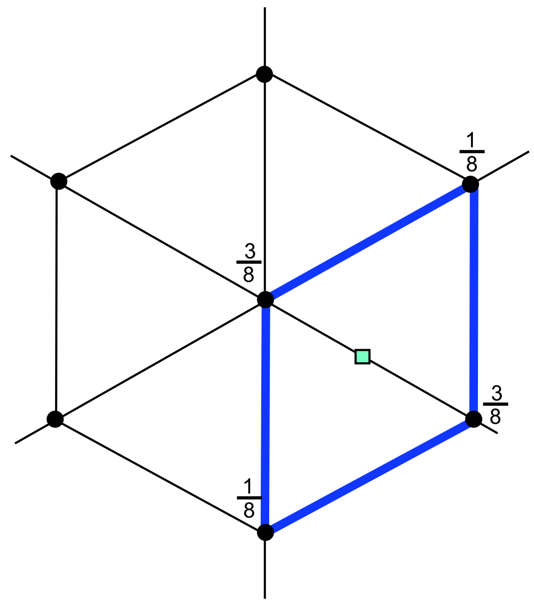
$$P_0^{j+1} = \frac{5}{8}P_0^j + \frac{1}{16}\sum_{i=1}^{6} P_i^j$$

- Edge rule:

$$P_i^{j+1} = \frac{3}{8}(P_0^j + P_i^j) + \frac{1}{8}(P_{i+1}^j + P_{i-1}^j)$$

where P_{i+1} and P_{i-1} just indicate the two neighbors of P_i.





Subdivision Matrix

The subdivision matrix (local) from the j level to $j+1$ level is

$$\begin{pmatrix} P_0^{j+1} \\ P_1^{j+1} \\ P_2^{j+1} \\ P_3^{j+1} \\ P_4^{j+1} \\ P_5^{j+1} \\ P_6^{j+1} \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 \\ \frac{3}{8} & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{3}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 \\ \frac{3}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} P_0^{j} \\ P_1^{j} \\ P_2^{j} \\ P_3^{j} \\ P_4^{j} \\ P_5^{j} \\ P_6^{j} \end{pmatrix}$$

Extraordinary Vertices

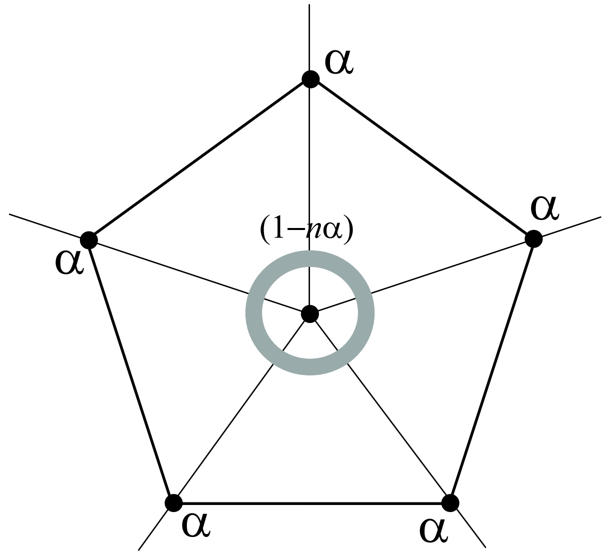
The case in which the neighbor's number is not 6 is called extraordinary. For this case, the edge rule is the same. The vertex rule can be naturally generalized as follows:

$$P_0^{j+1} = (1 - \alpha n) P_0^j + \alpha \sum_{i=1}^{n} P_i^j$$

where n is the number of neighbors and

$$\alpha = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2\right).$$

We can easily verify that in the case $n = 6$, $\alpha = \frac{1}{16}$.



At an extraordinary point, we need to alter our vertex subdivision rule.

Fractals

Consider a complex number $z = a + bi$ as a point (a, b) or vector in the Real Euclidean plane [1, i] with modulus $|z|$ the length of the vector and equal to $\sqrt{a^2 + b^2}$.

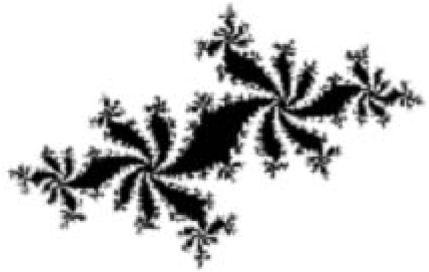
Complex arithmetic rules:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

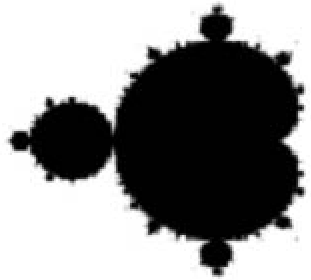
$z \rightarrow z^2$

All numbers with modulus 1 will stay at moduluas 1 and is the *attractor set* or *fixed-point* of this **iterated function system**.

Julia Set for the point c: The attractor set of the iterated function system $z \rightarrow z^2 + c$ with c a complex constant



Julia Set for $c = -0.62 - 0.44i$



Mandelbrot Set: Color the point c black if **Julia** (c) is connected, and *white* otherwise.

Fractal Dimension:

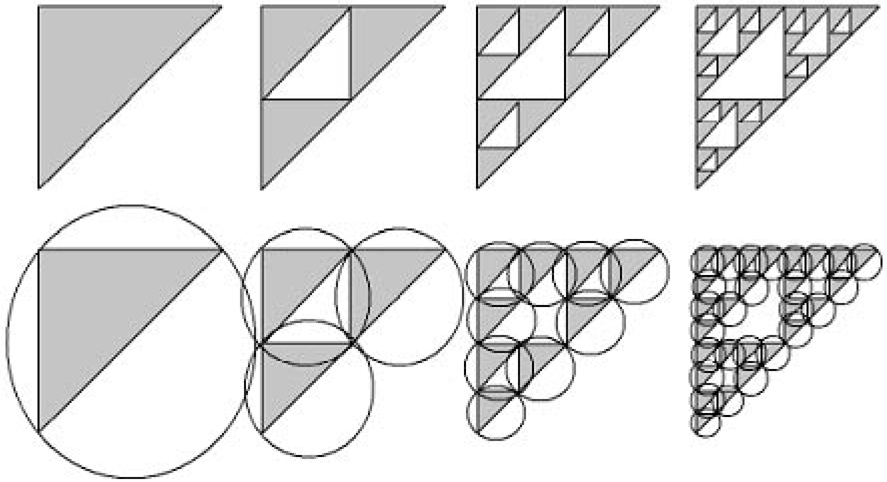
$N(A, \epsilon)$ = smallest number of ϵ-balls needed to cover A.

Object A has dimension d if $N(A, \epsilon)$ grows as $C(1/\epsilon)^d$ for constant C

$$\text{Fractal dimension } d = \lim_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)}$$

A **fractal** is an object which is *self-similar at different scales* and has a *non-integer fractal dimension*

$$\begin{aligned} d &= \lim_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)} \\ &= \lim_{k \to \infty} \frac{\ln N(A, (1/2^k))}{\ln(1/(1/2^k))} \\ &= \lim_{k \to \infty} \frac{\ln 3^k}{\ln 2^k} = \lim_{k \to \infty} \frac{k \ln 3}{k \ln 2} \\ &= \lim_{k \to \infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \approx 1.58496. \end{aligned}$$



The Sierpinki triangle covered by 3^k $(1/2^k)$-balls

Repeated Subdivision rule:

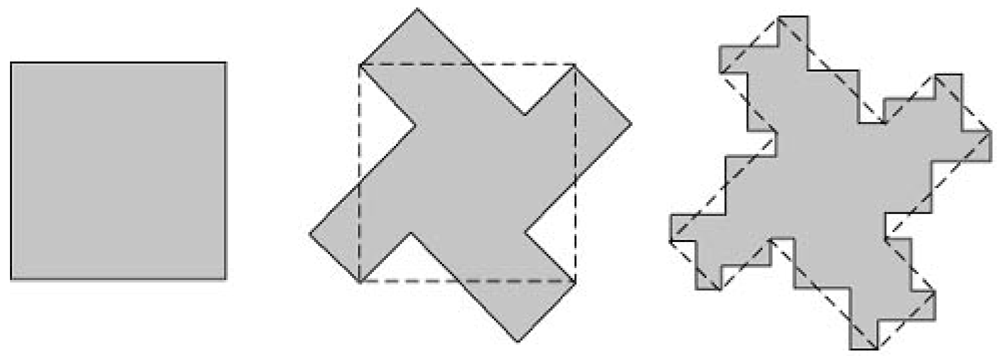
Replace each piece of length x by b nonoverlapping piece of length x/a.

Fractal dimension is

$$d = \frac{\ln b}{\ln a}$$

For object below the area doesn't change but boundary length does. The fractal dimension is

$$\frac{\ln 4}{\ln(2\sqrt{2})} = 0.8.$$



An object with a fractal boundary via repeated subdivision.