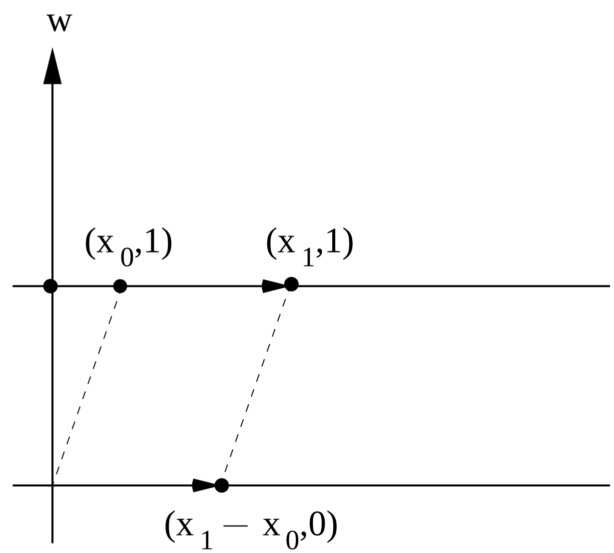
Vector and Affine Algebra

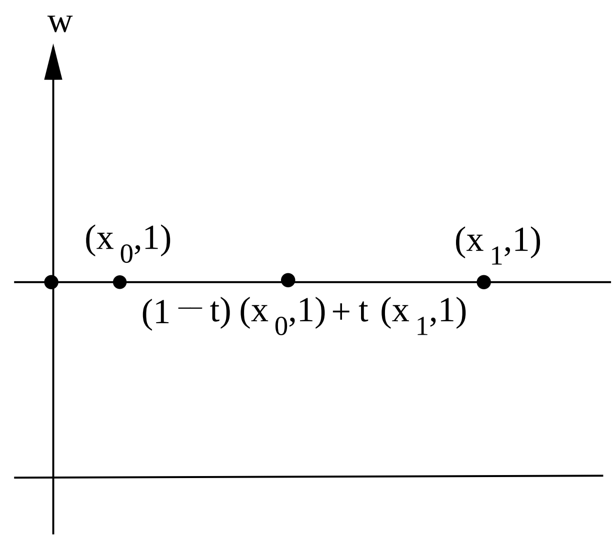
- Difference of points

$$(x_1, 1) - (x_0, 1) = (x_1 - x_0, 0)$$



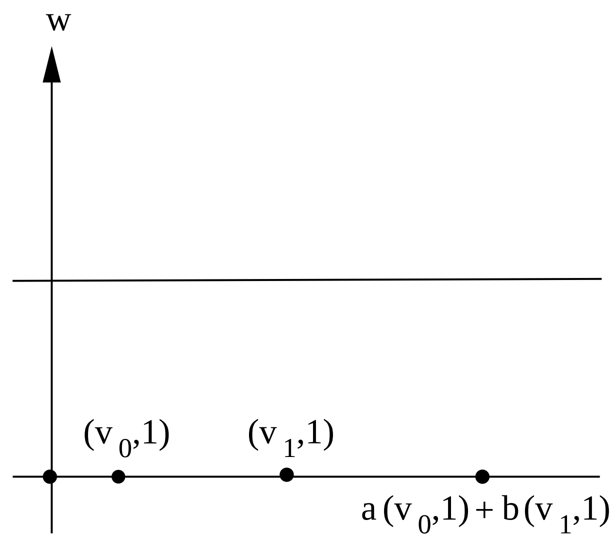
- Affine combination of points

$$(1-t)(x_1, 1) + t(x_0, 1) = ((1-t)x_1 + tx_0, 1)$$



- Linear combinations of vectors

$$a(v_0, 0) + b(v_1, 0) = (av_0 + bv_1, 0)$$



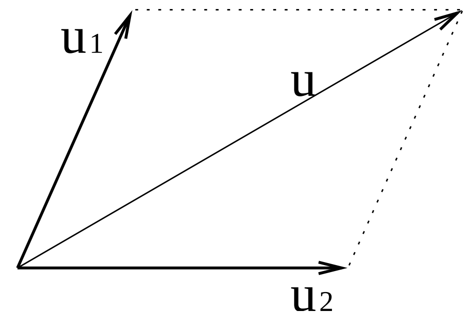
Linear Transformations

Vector space $\mathcal{V}$

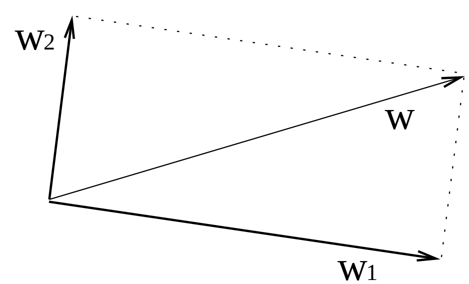
- Linear combinations of vectors in $\mathcal{V}$ are in $\mathcal{V}$
- For $\vec{u}, \vec{v} \in \mathcal{V}$
  - $\vec{u} + \vec{v} \in \mathcal{V}$
  - $\alpha\vec{u} \in \mathcal{V}$ for any scalar α
  - In general, $\sum_i \alpha_i \vec{u}_i \in \mathcal{V}$ for any scalars α_i
- Linear transformations
  - Let $\mathbf{T} : \mathcal{V}_0 \mapsto \mathcal{V}_1$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are vector spaces
  - Then $\mathbf{T}$ is *linear* iff
    - $\mathbf{T}(\vec{u} + \vec{v}) = \mathbf{T}(\vec{u}) + \mathbf{T}(\vec{v})$
    - $\mathbf{T}(\alpha\vec{u}) = \alpha\mathbf{T}(\vec{u})$
    - In general, $\mathbf{T}\left(\sum_i \alpha_i \vec{u}_i\right) = \sum_i \alpha_i \mathbf{T}\left(\vec{u}_i\right)$

Example of linear tranformation for vectors

$$u = \alpha_1 u_1 + \alpha_2 u_2$$



$$\begin{aligned} T(u) &= w \\ &= T(\alpha_1 u_1 + \alpha_2 u_2) \\ &= T(\alpha_1 u_1) + T(\alpha_2 u_2) \\ &= \alpha_1 T(u_1) + \alpha_2 T(u_2) \end{aligned}$$



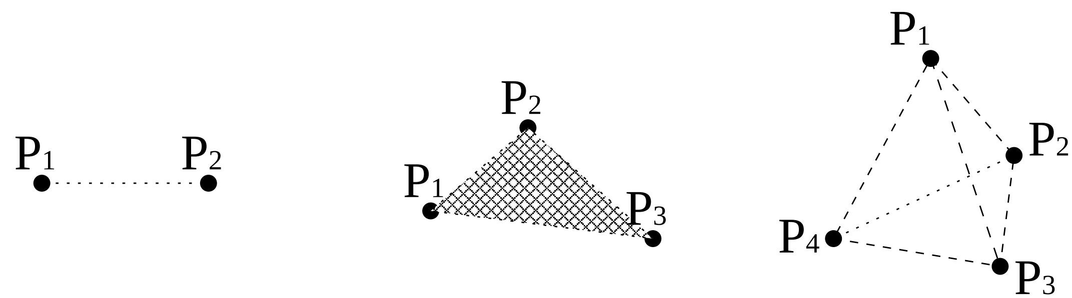
Affine Transformations

Affine space $\mathcal{A} = (\mathcal{V}, \mathcal{P})$

- For $\vec{u} \in \mathcal{V}$ and $P \in \mathcal{P}$
$$P + \vec{u} \in \mathcal{P}$$
- Define *point subtraction*:
  - For $P, Q \in \mathcal{P}$ and $\vec{u} \in \mathcal{V}$, if $P + \vec{u} = Q$, then $Q - P \equiv \vec{u}$
  - So in general we have $\sum_i \alpha_i P_i$ is a *vector* iff $\sum_i \alpha_i = 0$
- Define *point blending*:
  - For $P, P_1, P_2 \in \mathcal{P}$ and scalar α, if $P = P_1 + \alpha\,(P_2 - P_1)$ then $P \equiv (1 - \alpha)\,P_1 + \alpha P_2$
  - This can also be written $P \equiv \alpha_1 P_1 + \alpha_2 P_2$ where $\alpha_1 + \alpha_2 = 1$
  - So in general we have $\sum_i \alpha_i P_i$ is a *point* iff $\sum_i \alpha_i = 1$
- Geometrically, we have $\frac{|P - P_0|}{|P - P_1|} = \frac{d_1}{d_2}$ or $P = \frac{d_1 P_1 + d_2 P_2}{d_1 + d_2}$
- Vectors can always be combined linearly $\sum_i \alpha_i \vec{u}_i$
- Points can be combined linearly $\sum_i \alpha_i P_i$ iff
  - The coefficients sum to 1, giving a point ("affine combination")
  - The coefficients sum to 0, giving a vector ("vector combination")

– Example affine combination:

$$P(t) = P_0 + t(P_1 - P_0) = (1 - t)P_0 + tP_1$$



– This says any point on the line is an affine combination of the line segment's endpoints.

- Affine transformations
  – Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are affine spaces
  – $\mathbf{T}$ is said to be an *affine transformation* iff
    * $\mathbf{T}$ maps vectors to vectors and points to points
    * $\mathbf{T}$ is a linear transformation on the vectors
    * $\mathbf{T}(P + \vec{u}) = \mathbf{T}(P) + \mathbf{T}(\vec{u})$
  – Properties of affine transformations
    * $\mathbf{T}$ preserves affine combinations:

$$\mathbf{T}(\alpha_0 P_0 + \cdots + \alpha_n P_n) = \alpha_0 \mathbf{T}(P_0) + \cdots + \alpha_n \mathbf{T}(P_n)$$

where $\sum_i \alpha_i = 0$ or $\sum_i \alpha_i = 1$

* $\mathbf{T}$ maps lines to lines:

$$\mathbf{T}((1-t)P_0 + tP_1) = (1-t)\mathbf{T}(P_0) + t\mathbf{T}(P_1)$$

* $\mathbf{T}$ is affine iff it preserves ratios of distance along a line:

$$P = \frac{d_0 P_0 + d_1 P_1}{d_0 + d_1} \Rightarrow \mathbf{T}(P) = \frac{d_0 \mathbf{T}(P_0) + d_1 \mathbf{T}(P_1)}{d_0 + d_1}$$

* $\mathbf{T}$ maps parallel lines to parallel lines (can you prove this?)

– Example affine transformations
  * Rigid body motions (translations, rotations)
  * Scales, reflections
  * Shears

Matrix Representation of Transformations

- Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be affine spaces.
  Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ be an affine transformation.
  Let $F_0 = (\vec{i}_0, \vec{j}_0, \mathcal{O}_0)$ be a frame for $\mathcal{A}_0$.
  Let $F_1 = (\vec{i}_1, \vec{j}_1, \mathcal{O}_1)$ be a frame for $\mathcal{A}_1$.
- Let $P = x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0$ be a point in $\mathcal{A}_0$.
  The *coordinates* of P relative to $\mathcal{A}_0$ are $(x, y, 1)$.

  This can also be represented in vector form as $P = \begin{bmatrix} \vec{i}_0 & \vec{j}_0 & \mathcal{O}_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- What are the coordinates $(x', y', 1)$ of $\mathbf{T}(P)$ relative to F_1?
  - An affine transformation is characterized by the image of a frame in the domain.

$$\begin{aligned} \mathbf{T}(P) &= \mathbf{T}(x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0) \\ &= x\mathbf{T}(\vec{i}_0) + y\mathbf{T}(\vec{j}_0) + \mathbf{T}(\mathcal{O}_0) \end{aligned}$$

  - $\mathbf{T}(\vec{i}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{i}_0) = t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1$.
  - Likewise $\mathbf{T}(\vec{j}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{j}_0) = t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1$.
  - Finally $\mathbf{T}(\mathcal{O}_0)$ must be an affine combination of $\vec{i}_1$, $\vec{j}_1$, and $\mathcal{O}_1$, say $\mathbf{T}(\mathcal{O}_0) = t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1$.

– Then by substitution we get

$$\begin{aligned}
\mathbf{T}(P) &= x(t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1) + y(t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1) + t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1 \\
&= \begin{bmatrix} t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1 & t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1 \end{bmatrix} \; t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \vec{i}_1 \; \vec{j}_1 \; \mathcal{O}_1 \end{bmatrix} \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\end{aligned}$$

Using $\mathbf{M}_T$ to denote the matrix, we see that $F_0 = F_1\mathbf{M}_T$

- Let $\mathbf{T}(P) = P' = x'\vec{i}_1 + y'\vec{j}_1 + \mathcal{O}_1$

In vector form this is

$$P' = \left[\vec{i}_1\ \vec{j}_1\ \mathcal{O}_1\right] \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$= \left[\vec{i}_1\ \vec{j}_1\ \mathcal{O}_1\right] \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

So we see that

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can write this in shorthand – $\mathbf{p}' = \mathbf{M}_T\mathbf{p}$

- $\mathbf{M}_T$ is the *matrix representation* of **T**
  - The first column of $\mathbf{M}_T$ represents $\mathbf{T}(\vec{i}_0)$
  - The second column of $\mathbf{M}_T$ represents $\mathbf{T}(\vec{j}_0)$
  - The third column of $\mathbf{M}_T$ represents $\mathbf{T}(\mathcal{O}_0)$

- *Translation*
  - Points are transformed as $\left[x'\ y'\ 1\right]^T = [x\ y\ 1]^T + [\Delta x\ \Delta y\ 0]^T$.
  - Vectors don't change.
  - Thus translation is affine but not linear.
    If it were linear, we would have $\mathbf{T}(P+Q) = \mathbf{T}(P) + \mathbf{T}(Q)$, but point addition is undefined.
  - Translation can be applied to sums of vectors and vector-point sums.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $T(\Delta x, \Delta y)$

- *Scale*
  - Linear transform — applies equally to points and vectors
  - Points transform as $\left[x' \; y' \; 1\right]^T = [xS_x \; yS_y \; 1]^T$.
  - Vectors transform as $\left[x' \; y' \; 0\right]^T = [xS_x \; yS_y \; 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $S(S_x, S_y)$
  - Note that this is *origin sensitive*.
  - How do you do reflections?

- *Rotate*
  - Linear transform — applies equally to points and vectors
  - Points transform as
    $\left[x' \; y' \; 1\right]^T = [x\cos(\theta) - y\sin(\theta) \;\; x\sin(\theta) + y\cos(\theta) \;\; 1]^T.$
  - Vectors transform as
    $\left[x' \; y' \; 0\right]^T = [x\cos(\theta) - y\sin(\theta) \;\; x\sin(\theta) + y\cos(\theta) \;\; 0]^T.$
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $R(\theta)$
  - Note that this is *origin sensitive*.

- *Shear*
  - Linear transform — applies equally to points and vectors
  - Points transform as $\left[x'\ y'\ 1\right]^T = [x + \alpha y,\ y + \beta x,\ 1]^T$.
  - Vectors transform as $\left[x'\ y'\ 0\right]^T = [x + \alpha y,\ y + \beta x,\ 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $Sh(\alpha, \beta)$

- Composition of Transformations
  - Now we have some basic transformations, how do we create and represent arbitrary affine transformations?
  - We can derive an arbitrary affine transform as a sequence of basic transformations, then compose the transformations
  - Example — scaling about an arbitrary point $[x_c\ y_c\ 1]^T$
    1. Translate $[x_c\ y_c\ 1]^T$ to $[0\ 0\ 1]$ $(T(-x_c, -y_c))$
    2. Scale $\left[x'\ y'\ 1\right]^T = S(S_x, S_y)\,[x\ y\ 1]^T$
    3. Translate $[0\ 0\ 1]^T$ back to $[x_c\ y_c\ 1]$ $(T(x_c, y_c))$
  - The sequence of transformation steps is
    $T(-x_c, -y_c) \circ S(S_x, S_y) \circ T(x_c, y_c)$

– In matrix form this is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & x_c(1 - S_x) \\ 0 & S_y & y_c(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

– Note that the matrices are arranged from *right to left* in the order of the steps.

– The order is important (why)?

- Three Dimensional Transformations
  - A point is $\mathbf{p} = [x\ y\ z\ 1]$, a vector $\vec{v} = [x\ y\ z\ 0]$
  - Translation:

$$T(\Delta x, \Delta y, \Delta z) = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Scale:

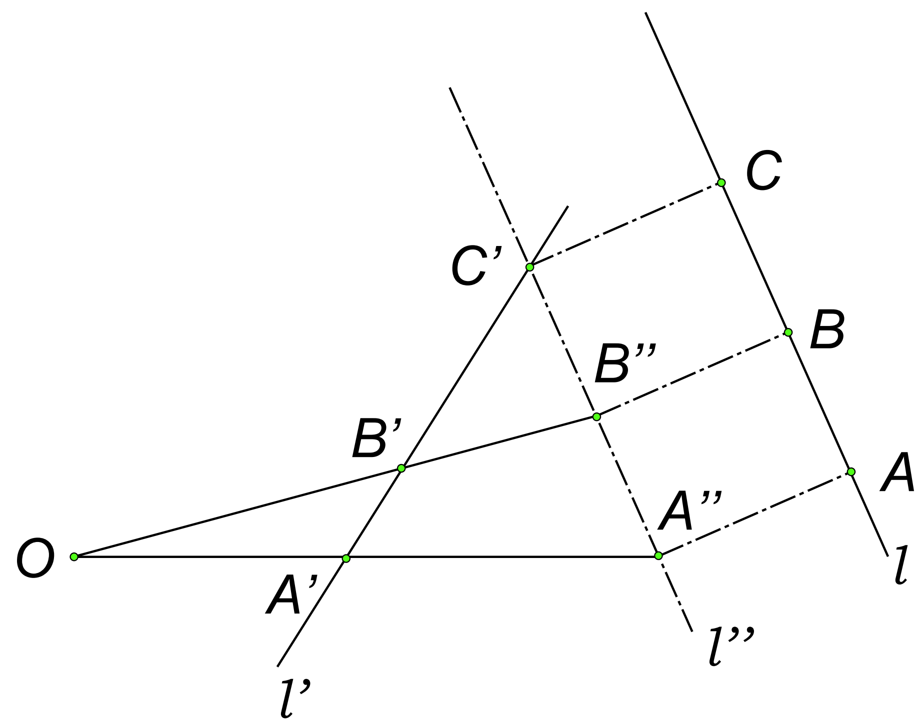
$$S(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Rotation:

$$R_z(\Theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Extra: Example of Invariance of Projective Transformation, The Cross Ratio

Definition:



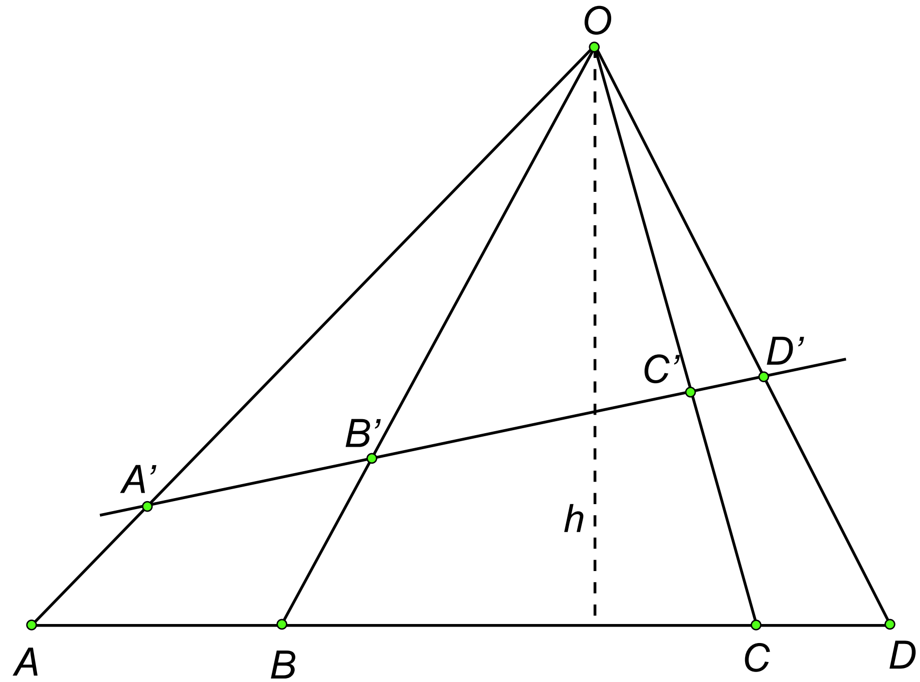
$$x = \frac{CA}{CB} \Bigg/ \frac{DA}{DB}$$

$$\frac{CA}{CB} \Bigg/ \frac{DA}{DB} = \frac{C'A'}{C'B'} \Bigg/ \frac{D'A'}{D'B'}$$

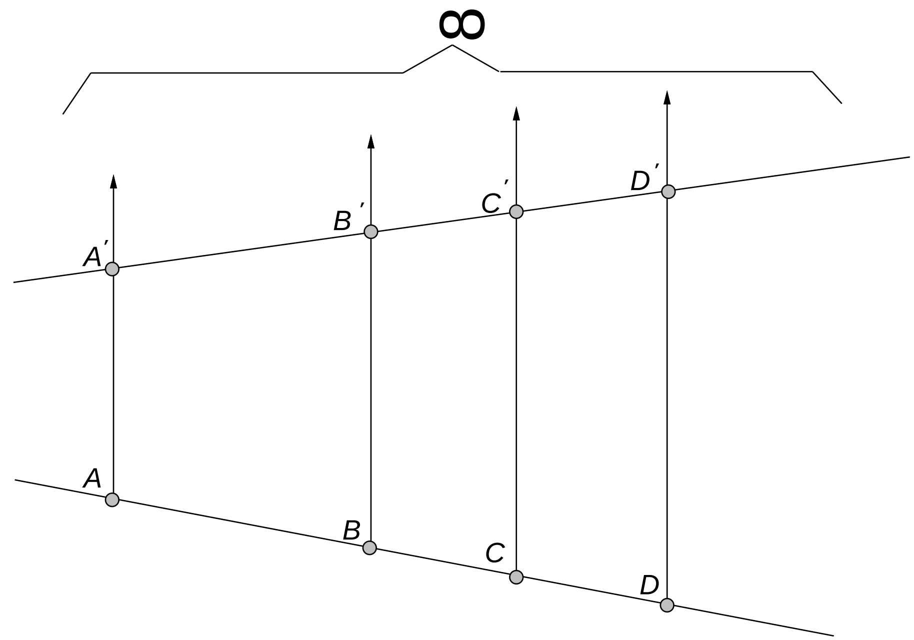
$$\text{area } OCA = \frac{1}{2} h \cdot CA = \frac{1}{2} OA \cdot OC \sin \angle COA$$
$$\text{area } OCB = \frac{1}{2} h \cdot CB = \frac{1}{2} OB \cdot OC \sin \angle COB$$
$$\text{area } ODA = \frac{1}{2} h \cdot DA = \frac{1}{2} OA \cdot OD \sin \angle DOA$$
$$\text{area } ODB = \frac{1}{2} h \cdot DB = \frac{1}{2} OB \cdot OD \sin \angle DOB$$

Hence

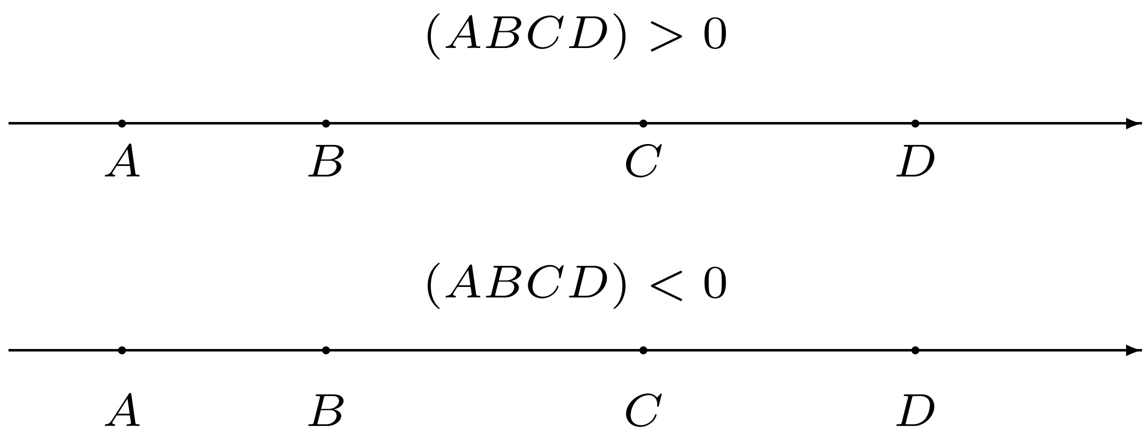
$$\frac{CA}{CB} \Big/ \frac{DA}{DB} = \frac{CA}{CB} \cdot \frac{DB}{DA} = \frac{OA \cdot OC \sin \angle COA}{OB \cdot OC \sin \angle COB} \cdot \frac{OB \cdot OD \sin \angle DOB}{OA \cdot OD \sin \angle DOA}$$
$$= \frac{\sin \angle COA}{\sin \angle COB} \cdot \frac{\sin \angle DOB}{\sin \angle DOA}$$



Invariance of cross-ratio under central projection



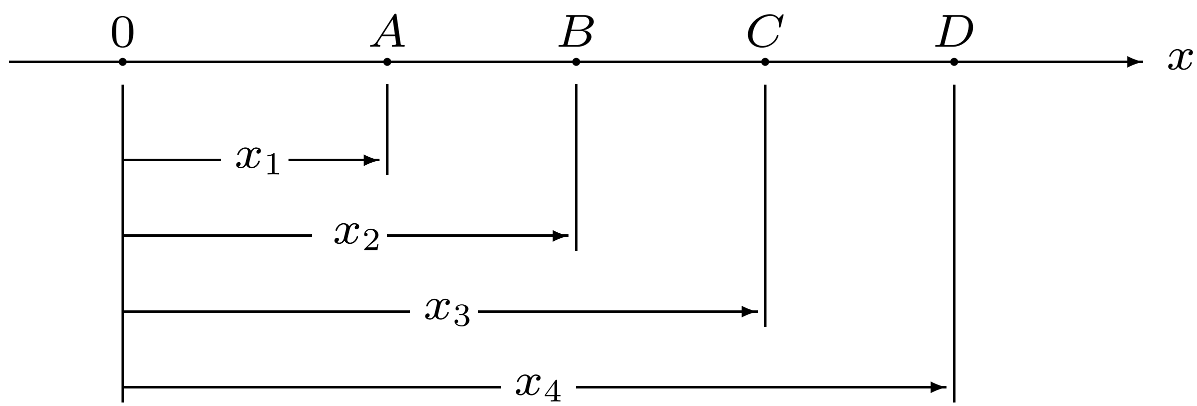
Invariance of cross-ratio under parallel projection



Sign of cross-ratio

$$(ABCD) = \frac{CA}{CB} \Big/ \frac{DA}{DB} = \frac{x_3 - x_1}{x_3 - x_2} \Big/ \frac{x_4 - x_1}{x_4 - x_2}$$

$$= \frac{x_3 - x_1}{x_3 - x_2} \cdot \frac{x_4 - x_2}{x_4 - x_1}$$



Cross-ratio in terms of coordinates.

Reading Assignment and News

Chapter 4 pages 181 - 201, of Recommended Text.

Please also track the News section of the Course Web Pages for the most recent Announcements related to this course.

(http://www.cs.utexas.edu/users/bajaj/graphics23/cs354/)