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EVENT: Start with the initial **nqthm** theory.

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; A Mechanical Checking of a Theorem About a Card Trick

;  Robert S. Boyer, May 22, 1991

; This is a formalization, in the Nqthm logic, of a card trick theorem
; that de Bruijn taught Huet, Huet taught Moore, and Moore taught me.
; Mine differs from a treatment of the same problem by Moore in that
; he uses oracles to simulate shuffling, whereas I use a merge
; predicate.

; Here is Moore's statement of the trick:

   ; Suppose you have a deck of cards of even length.  Suppose the cards
   ; alternate between red ones and black ones.  Cut the deck into two
   ; piles, a and b.  Shuffle a and b together.  Then the following is
   ; true of the shuffled deck.  If the bottom-most cards in a and b are
   ; of different color, then when the cards of the shuffled deck are
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   ; taken from the top in adjacent pairs, each pair contains a card of
   ; each color.  On the other hand, if the bottom-most cards in a and b
   ; are the same color, the above pairing property holds after rotating
   ; the shuffled deck by one card, i.e., moving the bottom card to the
   ; top.

#| For other references see

The Gilbreath Tick: A Case Study in Axiomatization and Proof Development in the
COQ Proof Assistant, Gerard Huet, Technical Report 1511, INRIA, September,
1991.

M. Gardner, Mathematical Recreation Column, Scientific American, Aug. 1960.,
p. 149, vol. 203, no. 2.

N. Gilbreath, Magnetic Colors, The Linking Ring, 38(5), 1958, p. 60.

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; Now on to my formalization.  We first define the six functions
; needed in the statement of the theorem.  The main, all encompassing,
; theorem is stated at the very end, and is named ``all''.

; Intuitively, we imagine that cards are arbitrary objects, but
; numbers are ``red'' and nonnumbers are ``black.''
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DEFINITION:
opposite-color $(x,\ y) = (((x \in \mathbf{N}) \wedge (y \notin \mathbf{N})) \vee ((y \in \mathbf{N}) \wedge (x \notin \mathbf{N})))$

DEFINITION:
alternating-colors (x)
$=$ **if** $(x \simeq \mathbf{nil}) \vee (\mathrm{cdr}\,(x) \simeq \mathbf{nil})$ **then t**
else opposite-color (car (x), cadr (x))
$\wedge$ alternating-colors (cdr (x)) **endif**

DEFINITION:
paired-colors (x)
$=$ **if** $(x \simeq \mathbf{nil}) \vee (\mathrm{cdr}\,(x) \simeq \mathbf{nil})$ **then t**
else opposite-color (car (x), cadr (x)) $\wedge$ paired-colors (cddr (x)) **endif**

DEFINITION:
plistp (x)

$=\quad$ **if** $x \simeq$ **nil then** $x =$ **nil**
else plistp (cdr (x)) **endif**

DEFINITION:
shufflep (x, y, z)
$=\quad$ **if** $z \simeq$ **nil then** ($x =$ **nil**) $\wedge$ ($y =$ **nil**) $\wedge$ ($z =$ **nil**)
elseif $x \simeq$ **nil then** ($x =$ **nil**) $\wedge$ ($y = z$) $\wedge$ plistp (y)
elseif $y \simeq$ **nil then** ($y =$ **nil**) $\wedge$ ($x = z$) $\wedge$ plistp (x)
else ((car (x) = car (z)) $\wedge$ shufflep (cdr (x), y, cdr (z)))
$\vee$ ((car (y) = car (z)) $\wedge$ shufflep (x, cdr (y), cdr (z))) **endif**

DEFINITION:
even-length (l)
$=\quad$ **if** $l \simeq$ **nil then t**
elseif cdr (l) $\simeq$ **nil then f**
else even-length (cddr (l)) **endif**

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; That ends the definitions needed to state the final theorem ``all,'' given
; at the end of this file.

; We now proceed to develop 17 auxiliary lemmas to help prove
; ``all.''
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THEOREM: al->pp
alternating-colors (x) $\rightarrow$ paired-colors (x)

DEFINITION:
silly (x, y, z)
$=\quad$ **if** $z \simeq$ **nil then t**
else list (silly (cddr (x), y, cddr (z)),
silly (cdr (x), cdr (y), cddr (z)),
silly (x, cddr (y), cddr (z))) **endif**

THEOREM: main
(shufflep (x, y, z)
$\wedge$ alternating-colors (x)
$\wedge$ alternating-colors (y)
$\wedge$ listp (x)
$\wedge$ listp (y)
$\wedge$ opposite-color (car (x), car (y)))
$\rightarrow$ paired-colors (z)

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; The closer to a gensym is the name of a lemma, the more boring and
; obvious it is.  As usual in such Nqthm proofs, these tedious lemmas
; are conceived by carefully reading failed Nqthm proofs rather than
; by thinking ahead.  The idea is to prove only those things which are
; suggested by Nqthm failures.
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THEOREM: f2
$(\text{plistp}\,(d) \land \text{plistp}\,(c)) \rightarrow \text{shufflep}\,(c,\ d,\ \text{append}\,(c,\ d))$

THEOREM: f3
$(\text{plistp}\,(d) \land \text{plistp}\,(c)) \rightarrow \text{shufflep}\,(c,\ d,\ \text{append}\,(d,\ c))$

THEOREM: cdr-append
$$\begin{aligned} & \text{cdr}\,(\text{append}\,(c,\ d)) \\ = \quad & \textbf{if}\ \text{listp}\,(c)\ \ \textbf{then}\ \text{append}\,(\text{cdr}\,(c),\ d) \\ & \textbf{else}\ \text{cdr}\,(d)\ \textbf{endif} \end{aligned}$$

THEOREM: f4
$(\text{plistp}\,(w) \land \text{shufflep}\,(x,\ y,\ z)) \rightarrow \text{shufflep}\,(x,\ \text{append}\,(y,\ w),\ \text{append}\,(z,\ w))$

THEOREM: f5
$(\text{plistp}\,(w) \land \text{shufflep}\,(x,\ y,\ z)) \rightarrow \text{shufflep}\,(\text{append}\,(x,\ w),\ y,\ \text{append}\,(z,\ w))$

THEOREM: trick
$$\begin{aligned} & (\text{listp}\,(x) \land \text{listp}\,(y) \land \text{shufflep}\,(x,\ y,\ z)) \\ \rightarrow \quad & (\text{shufflep}\,(\text{append}\,(\text{cdr}\,(x),\ \text{list}\,(\text{car}\,(x))),\ y,\ \text{append}\,(\text{cdr}\,(z),\ \text{list}\,(\text{car}\,(z)))) \\ & \ \lor\ \ \text{shufflep}\,(x, \\ & \qquad\qquad \text{append}\,(\text{cdr}\,(y),\ \text{list}\,(\text{car}\,(y))), \\ & \qquad\qquad \text{append}\,(\text{cdr}\,(z),\ \text{list}\,(\text{car}\,(z))))) \end{aligned}$$

THEOREM: car-append
$$\begin{aligned} & \text{car}\,(\text{append}\,(x,\ y)) \\ = \quad & \textbf{if}\ \text{listp}\,(x)\ \ \textbf{then}\ \text{car}\,(x) \\ & \textbf{else}\ \text{car}\,(y)\ \textbf{endif} \end{aligned}$$

THEOREM: f12
$$\begin{aligned} & (\text{alternating-colors}\,(\text{append}\,(c,\ \text{list}\,(d))) \land (\neg\ \text{opposite-color}\,(d,\ e))) \\ \rightarrow \quad & \text{alternating-colors}\,(\text{append}\,(c,\ \text{list}\,(e))) \end{aligned}$$

THEOREM: f6
$$\begin{aligned} & (\text{listp}\,(l) \land \text{even-length}\,(l) \land \text{alternating-colors}\,(l)) \\ \rightarrow \quad & \text{alternating-colors}\,(\text{append}\,(\text{cdr}\,(l),\ \text{list}\,(\text{car}\,(l)))) \end{aligned}$$

THEOREM: g19
$$\begin{aligned} & (\text{alternating-colors}\,(c) \land (\text{car}\,(c) \in \mathbf{N}) \land \text{even-length}\,(c) \land (v \in \mathbf{N})) \\ \rightarrow \quad & \text{alternating-colors}\,(\text{append}\,(c,\ \text{list}\,(v))) \end{aligned}$$

THEOREM: g20
$(\text{alternating-colors}\,(c) \wedge (\text{car}\,(c) \notin \mathbf{N}) \wedge \text{even-length}\,(c) \wedge (v \notin \mathbf{N}))$
$\rightarrow \quad \text{alternating-colors}\,(\text{append}\,(c,\ \text{list}\,(v)))$

THEOREM: fap
$(\text{shufflep}\,(x,\ y,\ z)$
$\wedge \quad \text{alternating-colors}\,(x)$
$\wedge \quad \text{alternating-colors}\,(y)$
$\wedge \quad \text{even-length}\,(x)$
$\wedge \quad \text{even-length}\,(y)$
$\wedge \quad (\neg\ \text{opposite-color}\,(\text{car}\,(x),\ \text{car}\,(y))))$
$\rightarrow \quad \text{paired-colors}\,(\text{append}\,(\text{cdr}\,(z),\ \text{list}\,(\text{car}\,(z))))$

THEOREM: al2
$(\text{alternating-colors}\,(\text{append}\,(x,\ y))$
$\wedge \quad \text{even-length}\,(\text{append}\,(x,\ y))$
$\wedge \quad (\neg\ \text{opposite-color}\,(\text{car}\,(x),\ \text{car}\,(y))))$
$\rightarrow \quad (\text{even-length}\,(x) \wedge \text{even-length}\,(y))$

THEOREM: g16
$(\neg\ \text{alternating-colors}\,(x)) \rightarrow (\neg\ \text{alternating-colors}\,(\text{append}\,(x,\ y)))$

THEOREM: g17
$(\neg\ \text{alternating-colors}\,(y)) \rightarrow (\neg\ \text{alternating-colors}\,(\text{append}\,(x,\ y)))$

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; That ends the intermediate development of auxiliary lemmas.  We are now
; ready for the main result.
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THEOREM: all
$(\text{alternating-colors}\,(\text{append}\,(x,\ y))$
$\wedge \quad \text{even-length}\,(\text{append}\,(x,\ y))$
$\wedge \quad \text{shufflep}\,(x,\ y,\ z)$
$\wedge \quad \text{listp}\,(x)$
$\wedge \quad \text{listp}\,(y))$
$\rightarrow \quad \textbf{if}\ \text{opposite-color}\,(\text{car}\,(x),\ \text{car}\,(y))\ \ \textbf{then}\ \text{paired-colors}\,(z)$
$\qquad \textbf{else}\ \text{paired-colors}\,(\text{append}\,(\text{cdr}\,(z),\ \text{list}\,(\text{car}\,(z))))\ \textbf{endif}$

```
; To run this list of events takes about 1 hour of cpu time on a
; Sun-3/280.  To develop this list of events took about 15 hours of
; work.
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