#|

|#

EVENT: Start with the initial **nqthm** theory.

```
;;;;;;;;;;;;;;;;;;;;;;   mutex-atomic.ev    ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;  com.ev  ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;*sequence and finite set utilities

;;;The ith entry in l.
```

DEFINITION:
nth $(l,\ i)$
$=$ **if** listp (l)

then if $i = \texttt{1}$ **then** $\text{car}\,(l)$
else $\text{nth}\,(\text{cdr}\,(l),\ i - 1)$ **endif**
elseif $l \in \mathbf{N}$
then if $i = \texttt{1}$ **then** l
else f endif
else f endif

EVENT: Disable nth.

```
;;;update ith entry of l to be k
```

DEFINITION:
$\text{move}\,(l,\ i,\ k)$
$=$ **if** $i = \texttt{0}$ **then** l
elseif $l \simeq$ **nil**
then if $i = \texttt{1}$ **then** k
else l **endif**
elseif $i = \texttt{1}$ **then** $\text{cons}\,(k,\ \text{cdr}\,(l))$
else $\text{cons}\,(\text{car}\,(l),\ \text{move}\,(\text{cdr}\,(l),\ i - 1,\ k))$ **endif**

EVENT: Disable move.

DEFINITION: $\text{at}\,(l,\ i,\ k) = (\text{nth}\,(l,\ i) = k)$

EVENT: Disable at.

DEFINITION:
$\text{length}\,(l)$
$=$ **if** $\text{listp}\,(l)$ **then** $1 + \text{length}\,(\text{cdr}\,(l))$
else ZERO **endif**

EVENT: Disable length.

```
;;;The nth entry in l is in the list i.
```

DEFINITION: $\text{union-at-n}\,(l,\ n,\ i) = (\text{nth}\,(l,\ n) \in i)$

EVENT: Disable union-at-n.

```
;;;Any entry in l is in the list i.
```

DEFINITION:
$\text{all-union}\,(l,\ n,\ i)$
$=$ **if** $n \simeq \texttt{0}$ **then t**
else $\text{union-at-n}\,(l,\ n,\ i) \wedge \text{all-union}\,(l,\ n - 1,\ i)$ **endif**

EVENT: Disable all-union.

```
;;;There exists an entry in l which belongs to
;;;the list i, moreover when exists, some such
;;;j is returned.
```

DEFINITION:
exist-union (l, n, i)
$=$ **if** $n \simeq 0$ **then f**
elseif union-at-n (l, n, i) **then** n
else exist-union $(l, n - 1, i)$ **endif**

EVENT: Disable exist-union.

```
;;;n is in the intersection of l8-12 and g34.
```

DEFINITION:
intersect-8-12-3-4-at-n (n, l, g)
$=$ (union-at-n $(l, n,$ `'(8 9 10 11 12)`$)$ $\land$ union-at-n $(g, n,$ `'(3 4)`$))$

EVENT: Disable intersect-8-12-3-4-at-n.

```
;;;There exists n in the intersection of l8-12 and g34.
```

DEFINITION:
exist-intersect-8-12-3-4 (n, l, g)
$=$ **if** $n \simeq 0$ **then f**
elseif intersect-8-12-3-4-at-n (n, l, g) **then** n
else exist-intersect-8-12-3-4 $(n - 1, l, g)$ **endif**

EVENT: Disable exist-intersect-8-12-3-4.

```
;*Flag invariant.
```

DEFINITION:
lg-1-at-n (n, l, g)
$=$ ((at $(l, n, 0)$ $\land$ at $(g, n, 0)$)
$\lor$ (at $(l, n, 1)$ $\land$ at $(g, n, 0)$)
$\lor$ (at $(l, n, 2)$ $\land$ at $(g, n, 0)$)
$\lor$ (at $(l, n, 3)$ $\land$ at $(g, n, 1)$)
$\lor$ (at $(l, n, 4)$ $\land$ at $(g, n, 1)$))

EVENT: Disable lg-1-at-n.

DEFINITION:
lg-2-at-n $(n,\ l,\ g)$
$$= \quad ((\text{at}\,(l,\ n,\ \texttt{5}) \wedge \text{at}\,(g,\ n,\ \texttt{3})) \vee (\text{at}\,(l,\ n,\ \texttt{6}) \wedge \text{at}\,(g,\ n,\ \texttt{3})) \vee (\text{at}\,(l,\ n,\ \texttt{7}) \wedge \text{at}\,(g,\ n,\ \texttt{2})) \vee (\text{at}\,(l,\ n,\ \texttt{8}) \wedge \text{at}\,(g,\ n,\ \texttt{3})) \vee (\text{at}\,(l,\ n,\ \texttt{8}) \wedge \text{at}\,(g,\ n,\ \texttt{2})))$$

EVENT: Disable lg-2-at-n.

DEFINITION:
lg-3-at-n $(n,\ l,\ g)$
$$= \quad ((\text{at}\,(l,\ n,\ \texttt{9}) \wedge \text{at}\,(g,\ n,\ \texttt{4})) \vee (\text{at}\,(l,\ n,\ \texttt{10}) \wedge \text{at}\,(g,\ n,\ \texttt{4})) \vee (\text{at}\,(l,\ n,\ \texttt{11}) \wedge \text{at}\,(g,\ n,\ \texttt{4})) \vee (\text{at}\,(l,\ n,\ \texttt{12}) \wedge \text{at}\,(g,\ n,\ \texttt{4})))$$

EVENT: Disable lg-3-at-n.

DEFINITION:
lg-at-n $(n,\ l,\ g)$
$$= \quad (\text{lg-1-at-n}\,(n,\ l,\ g) \wedge \text{lg-2-at-n}\,(n,\ l,\ g) \wedge \text{lg-3-at-n}\,(n,\ l,\ g))$$

EVENT: Disable lg-at-n.

DEFINITION:
lg $(n,\ l,\ g)$
= **if** $n \simeq \texttt{0}$ **then t**
else lg-at-n $(n,\ l,\ g) \wedge \text{lg}\,(n - 1,\ l,\ g)$ **endif**

EVENT: Disable lg.

```
;*The set {1...n}.
```

DEFINITION:
nset (n)
= **if** $n \simeq \texttt{0}$ **then nil**
else cons $(n,\ \text{nset}\,(n - 1))$ **endif**

EVENT: Disable nset.

```
;;;n belongs to nset.
```

THEOREM: n-in-nset
$(n \not\simeq 0) \rightarrow (n \in \text{nset}(n))$

```
;;;Any element in nset is a number.
```

THEOREM: nset-number
$(k \in \text{nset}(n)) \rightarrow (k \in \mathbf{N})$

```
;;;If a nonzero number plus one belongs to nset,
;;;then so does the nonzero number itself.
```

THEOREM: add1-nset
$((k \not\simeq 0) \wedge ((1 + k) \in \text{nset}(n))) \rightarrow (k \in \text{nset}(n))$

```
;;;Any list has its length at least nonzero.
```

THEOREM: list-ln
$\text{listp}(l) \rightarrow (\text{length}(l) \neq 0)$

```
;;;(move l k i) is again a list if l is a list.
```

THEOREM: move-is-list
$\text{listp}(l) \rightarrow \text{listp}(\text{move}(l,\ k,\ i))$

EVENT: Enable length.

```
;;;(move l k i) has i as its kth entry.
;;;(enable length) is critical to prove this lemma.
```

THEOREM: move-nth
$(\text{listp}(l) \wedge (k \in \text{nset}(\text{length}(l)))) \rightarrow (\text{nth}(\text{move}(l,\ k,\ i),\ k) = i)$

THEOREM: zero-not-member-nset
$0 \notin \text{nset}(n)$

```
;;;Lists l and (move l k i) have the same length.
```

THEOREM: move-unchange-length
$(\text{listp}(l) \wedge (k \in \text{nset}(\text{length}(l))))$
$\rightarrow \quad (\text{length}(\text{move}(l,\ k,\ i)) = \text{length}(l))$

```
;;;Lists l and (move l k i) have the same entries
;;;except kth one.
```

THEOREM: move-unchange-other-than-nth
$(\text{listp}(l) \wedge (k \in \text{nset}(\text{length}(l))) \wedge (j \neq k))$
$\rightarrow \quad (\text{nth}(\text{move}(l,\ k,\ i),\ j) = \text{nth}(l,\ j))$

THEOREM: member-ex-union
exist-union $(l,\ n,\ i) \rightarrow$ (exist-union $(l,\ n,\ i) \in$ nset (n))

```
;;;(exist-union l n i) is a number.
```

THEOREM: number-ex-union
exist-union $(l,\ n,\ i) \rightarrow$ (exist-union $(l,\ n,\ i) \in \mathbf{N}$)

```
;;;(exist-intersect-8-12-3-4 n l g) belongs to nset.
```

THEOREM: member-intersect
exist-intersect-8-12-3-4 $(n,\ l,\ g)$
$\rightarrow$ (exist-intersect-8-12-3-4 $(n,\ l,\ g) \in$ nset (n))

```
;;;(exist-intersect-8-12-3-4 n l g) is a number.
```

THEOREM: number-intersect
exist-intersect-8-12-3-4 $(n,\ l,\ g) \rightarrow$ (exist-intersect-8-12-3-4 $(n,\ l,\ g) \in \mathbf{N}$)

```
;;;any member of nset is nonzero.
```

THEOREM: k-not-0
$(k \in$ nset $(n)) \rightarrow (k \not\simeq \texttt{0})$

```
;*lemmas for a0

;;;If j's entry in l is between 8..12 then
;;;(exist-union l n '(8 9 10 11 12)) holds.
```

THEOREM: j-ex-l8-12
$((j \in$ nset $(n)) \land$ union-at-n $(l,\ j,$ `'(8 9 10 11 12)`$))$
$\rightarrow$ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$)$

```
;;;Witness of (exist-union lp n '(8 9 10 11 12))
;;;has in lp its entry between 8...12.
```

THEOREM: ex-lp8-12-in-lp8-12
exist-union $(lp,\ n,$ `'(8 9 10 11 12)`$)$
$\rightarrow$ union-at-n $(lp,$
exist-union $(lp,\ n,$ `'(8 9 10 11 12)`$)$,
`'(8 9 10 11 12)`$)$

```
;;;If (not (exist-union l n '(8 9 10 11 12)))
;;;holds, then (not (exist-union g n '(4))) by lg.
```

THEOREM: ex-if4
$((\neg$ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$)) \land$ lg $(n,\ l,\ g))$
$\rightarrow$ $(\neg$ exist-union $(g,\ n,$ `'(4)`$))$

```
;;;If (not (exist-union g n '(1))) holds,
;;; then there is no entry either 3 or 4.
```

THEOREM: l34-empty
$((j \in \text{nset}(n)) \wedge \text{lg}(n, l, g) \wedge (\neg\ \text{exist-union}(g, n, \texttt{'(1)})))$
$\rightarrow \quad (\neg\ \text{union-at-n}(l, j, \texttt{'(3 4)}))$

```
;;;If j's entry in lp is 4, then (certainly)
;;;it is either 3 or 4.
```

THEOREM: lp4-then-un34
$\text{at}(lp, j, 4) \rightarrow \text{union-at-n}(lp, j, \texttt{'(3 4)})$

```
;;;If (exist-intersect-8-12-3-4 n l g) holds,
;;;then so does (exist-union g n '(3 4)).
```

THEOREM: int-8-12-3-4-then-un34
$\text{exist-intersect-8-12-3-4}(n, l, g) \rightarrow \text{exist-union}(g, n, \texttt{'(3 4)})$

```
;*lemmas for a1

;;;i is the witness of
;;;(exist-intersect-8-12-3-4 n lp gp).
```

THEOREM: int-wtn
$((j \in \text{nset}(n)) \wedge \text{intersect-8-12-3-4-at-n}(j, lp, gp))$
$\rightarrow \quad \text{exist-intersect-8-12-3-4}(n, lp, gp)$

```
;;;If there exists j such that j's entry in lp
;;;is between 8..12 and entry in gp is either 3 or 4
;;;then (intersect-8-12-3-4-at-n j lp gp) holds.
```

THEOREM: un8-12-and-un34-then-int
$(\text{union-at-n}(lp, j, \texttt{'(8 9 10 11 12)}) \wedge \text{union-at-n}(gp, j, \texttt{'(3 4)}))$
$\rightarrow \quad \text{intersect-8-12-3-4-at-n}(j, lp, gp)$

```
;;;By the two lemmas above,
;;;(exist-intersect-8-12-3-4 n lp gp) holds provided
;;;that there exists j such that j's entry in lp is
;;;between 8..12 and entry in gp is either 3 or 4.

;* ep-l8-12

;;;If the k's entry in l is 5, then the k's entry
;;;in g is 3 by lg.
```

THEOREM: lg-l5-g3
$((k \in \text{nset}(n)) \land \text{lg}(n, l, g) \land \text{at}(l, k, \texttt{5})) \rightarrow \text{at}(g, k, \texttt{3})$

```
;;;If the k's entry in gp is 3 then certainly
;;;it is either 3 or 4.
```

THEOREM: gp3-then-un34
$\text{at}(gp, k, \texttt{3}) \rightarrow \text{union-at-n}(gp, k, \texttt{'(3 4)})$

```
;;;nep-l8-12

;;;If the k's entry in l is between 8..12 then
;;;it is either between 8..11 or equal to 12.
```

THEOREM: case-k
$(\text{union-at-n}(l, k, \texttt{'(8 9 10 11 12)})$
$\land \quad (\neg \text{union-at-n}(l, k, \texttt{'(8 9 10 11)})))$
$\rightarrow \quad \text{at}(l, k, \texttt{12})$

```
;;;;;k-not-l8-12

;;;If (exist-intersect-8-12-3-4 n l g) holds
;;;then the witness has its entry in g either equal
;;;to 3 or 4.
```

THEOREM: intersect-8-12-3-4-then-3-4
$\text{exist-intersect-8-12-3-4}(n, l, g)$
$\rightarrow \quad \text{union-at-n}(g, \text{exist-intersect-8-12-3-4}(n, l, g), \texttt{'(3 4)})$

```
;;;If (exist-intersect-8-12-3-4 n l g) holds,
;;; then the witness has its entry in g between 8 and 12.
```

THEOREM: intersect-8-12-3-4-then-8-12
$\text{exist-intersect-8-12-3-4}(n, l, g)$
$\rightarrow \quad \text{union-at-n}(l, \text{exist-intersect-8-12-3-4}(n, l, g), \texttt{'(8 9 10 11 12)})$

```
;;;k-in-l8-11

;;;If k's entry in lp is between 9 and 12,
;;;then it is certainly between 8 and 12.
```

THEOREM: un9-12-then-un8-12
$\text{union-at-n}(lp, k, \texttt{'(9 10 11 12)})$
$\rightarrow \quad \text{union-at-n}(lp, k, \texttt{'(8 9 10 11 12)})$

```
;;;If the i's entry in l is between 9 and 12,
;;;then the k's entry in g is 4.
```

THEOREM: if4
$((j \in \text{nset}(n)) \land \text{lg}(n, l, g) \land \text{union-at-n}(l, j, \texttt{'(9 10 11 12)}))$
$\rightarrow \quad \text{at}(g, j, 4)$

```
;;;k-in-l12


;;;If (exist-union lp n '(8 9 10 11 12)) holds then
;;;its witness does not have its entry in lp equal to 1.
```

THEOREM: ex-lp8-12-not-in-lp0
$\text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)})$
$\rightarrow \quad (\neg\ \text{at}(lp, \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)}), 0))$

```
;;;If k's entry in lp is between 8 and 12,
;;; then it is either between 8 and 11 or 12.
```

THEOREM: k-in-lp9-12-or-lp8
$(\text{union-at-n}(lp, k, \texttt{'(8 9 10 11 12)})$
$\land \quad (\neg\ \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)})))$
$\rightarrow \quad \text{at}(lp, k, 8)$

```
;;;If the k's entry is either 5 or 7,
;;;then it is between 5 and 7.
```

THEOREM: un57-then-un5-12
$\text{union-at-n}(l, k, \texttt{'(5 7)}) \rightarrow \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11 12)})$

```
;;;If the k's entry in l is between 8 and 11,
;;;then it is between 5 and 12.
```

THEOREM: un8-11-then-un5-12
$\text{union-at-n}(l, k, \texttt{'(8 9 10 11)})$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11 12)})$

```
;;;If the k's entry in l is between 8 and 12,
;;;then it is between 5 and 12.
```

THEOREM: un8-12-then-un5-12
$\text{union-at-n}(l, k, \texttt{'(8 9 10 11 12)})$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11 12)})$

```
;*lemmas for a2


;;;i-eq-k-j-neq-k


;;;If the k's entry in l is either 10 or 11,
;;;then the k's entry in l is between 10 and 12.
```

THEOREM: un10-11-then-un10-12
union-at-n $(l, k,$ `'(10 11)`$) \rightarrow$ union-at-n $(l, k,$ `'(10 11 12)`$)$

```
;;;If the j's entry in g is either 0 or 1 then
;;;the j's entry in l is not between 5 and 12.
```

THEOREM: if1
$((j \in$ nset $(n)) \land$ lg $(n, l, g) \land$ union-at-n $(g, j,$ `'(0 1)`$))$
$\rightarrow$ $(\neg$ union-at-n $(l, j,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;j-eq-k-i-neq-k

;;;If the k's entry in l is between 5 and 7,
;;;then it is certainly between 5 and 12.
```

THEOREM: un5-7-then-un5-11
union-at-n $(l, k,$ `'(5 6 7)`$) \rightarrow$ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11)`$)$

```
;;;If the k's entry in lp is between 5 and 7 then
;;;it is certain between 5 and 11.
```

THEOREM: un57-then-un5-11
union-at-n $(l, k,$ `'(5 7)`$) \rightarrow$ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11)`$)$

```
;;;If the k's entry in l is between 8 and 11,
;;;then it is certainly between 5 and 11.
```

THEOREM: un8-11-then-un5-11
union-at-n $(l, k,$ `'(8 9 10 11)`$)$
$\rightarrow$ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11)`$)$

```
;;;If the k's entry in lp is between 5 and 12 and
;;;the k's entry in lp is between 5 and 7, then
;;;the k's entry in lp in fact is between 9 and 12.
```

THEOREM: k-in-lp5-7-or-lp8-or-lp9-12
(union-at-n $(lp, k,$ `'(5 6 7 8 9 10 11 12)`$)$
$\land$ $(\neg$ union-at-n $(lp, k,$ `'(5 6 7)`$))$
$\land$ $(\neg$ at $(lp, k, 8)))$
$\rightarrow$ union-at-n $(lp, k,$ `'(9 10 11 12)`$)$

```
;;;If the k's entry in l is between 5 and 11,
;;; then it is certainly between 5 and 12.
```

THEOREM: un5-11-then-un5-12
union-at-n $(l, k,$ `'(5 6 7 8 9 10 11)`$)$
$\rightarrow$ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11 12)`$)$

```
;;;If the k's entry in l is between 10 and 12,
;;; then it is certainly between 8 and 12.
```

Theorem: un10-12-then-un8-12
union-at-n$(l,\ i,$ `'(10 11 12)`$) \rightarrow$ union-at-n$(l,\ i,$ `'(8 9 10 11 12)`$)$

```
;;;j-eq-k-i-neq-k

;;;If (exist-union l n '(8 9 10 11 12)) does not hold,
;;;then the i's entry in l is not between 10 and 12.
```

Theorem: i-not-l10-12
$((i \in \text{nset}(n)) \wedge (\neg$ exist-union$(l,\ n,$ `'(8 9 10 11 12)`$)))$
$\rightarrow \quad (\neg$ union-at-n$(l,\ i,$ `'(10 11 12)`$))$

```
;*lemmas for a3

;;;j-eq-k-i-neq-k

;;;If the k's entry in l is between 5 and 11,
;;;then the k's entry in l is between 9 and 11.
```

Theorem: un5-11-eq-un58-or-un8-11
(union-at-n$(l,\ k,$ `'(5 6 7 8 9 10 11)`$)$
$\wedge \quad (\neg$ union-at-n$(l,\ k,$ `'(5 6 7 8)`$)))$
$\rightarrow \quad$ union-at-n$(l,\ k,$ `'(9 10 11)`$)$

```
;;;If the k's entry in g is 4,
;;;then the k's entry in l is between 5 and 8.
```

Theorem: a3-if4
$((k \in \text{nset}(n)) \wedge \text{lg}(n,\ l,\ g) \wedge \text{at}(g,\ k,\ 4))$
$\rightarrow \quad (\neg$ union-at-n$(l,\ k,$ `'(5 6 7 8)`$))$

```
;;;If the k's entry in l is between 5 and 11,
;;;and the k's entry in l is between 5 and 12,
;;;then the k's entry in l is 9 and 11.
```

Theorem: k-in-l5-11-g4-then-l9-11
$((k \in \text{nset}(n))$
$\wedge \quad \text{lg}(n,\ l,\ g)$
$\wedge \quad$ union-at-n$(l,\ k,$ `'(5 6 7 8 9 10 11)`$)$
$\wedge \quad \text{at}(g,\ k,\ 4))$
$\rightarrow \quad$ union-at-n$(l,\ k,$ `'(9 10 11)`$)$

```
; dropped second use hint -- rsb

;;;If the i's entry in l is 12,
;;;then the i's entry in l is between 8 and 12.
```

THEOREM: l12-then-un8-12
$\text{at}\,(l,\ i,\ \texttt{12}) \rightarrow \text{union-at-n}\,(l,\ i,\ \texttt{'(8 9 10 11 12)})$

```
;;;If (exist-union l n '(8 9 10 11 12)) does not hold,
;;;then the i's entry in l is 12.
```

THEOREM: i-not-in-l12
$((i \in \text{nset}\,(n)) \wedge (\neg\ \text{exist-union}\,(l,\ n,\ \texttt{'(8 9 10 11 12)})))$
$\rightarrow \quad (\neg\ \text{at}\,(l,\ i,\ \texttt{12}))$

```
;;;j-neq-k-i-eq-k

;;;If the k's entry in l is 11,
;;; then the k's entry in l is between 10 and 12.
```

THEOREM: l11-then-un10-12
$\text{at}\,(l,\ k,\ \texttt{11}) \rightarrow \text{union-at-n}\,(l,\ k,\ \texttt{'(10 11 12)})$

```
;;;If the j's entry in g is either 2 or 3,
;;;then the j's entry in l is between 5 and 8 by lg.
```

THEOREM: if3
$((j \in \text{nset}\,(n)) \wedge \text{lg}\,(n,\ l,\ g) \wedge (\neg\ \text{union-at-n}\,(g,\ j,\ \texttt{'(2 3)})))$
$\rightarrow \quad (\neg\ \text{union-at-n}\,(l,\ j,\ \texttt{'(5 6 7 8)}))$

```
;;;If the j's entry in l is between 5 and 12 and
;;;the j's entry in l is between 5 and 8, then
;;;the j's entry in l is 9 and 12.
```

THEOREM: l5-12-eq-l5-8-or-l9-12
$(\text{union-at-n}\,(l,\ j,\ \texttt{'(5 6 7 8 9 10 11 12)})$
$\wedge \quad (\neg\ \text{union-at-n}\,(l,\ j,\ \texttt{'(5 6 7 8)})))$
$\rightarrow \quad \text{union-at-n}\,(l,\ j,\ \texttt{'(9 10 11 12)})$

```
;;;i-j-eq-k

;;;If the k's entry in lp is 12,
;;;then it is certainly between 5 and 12.
```

THEOREM: l12-then-un9-12
$\text{at}\,(lp,\ k,\ \texttt{12}) \rightarrow \text{union-at-n}\,(lp,\ k,\ \texttt{'(9 10 11 12)})$

```
;*lemmas for b1a

;;;If the u's entry in g is 4,
;;;then the u's entry in l is between 8 and 12 by lg.
```

THEOREM: b1a-if4
$((u \in \text{nset}(n)) \land \text{lg}(n, l, g) \land \text{at}(g, u, 4))$
$\rightarrow \quad \text{union-at-n}(l, u, \texttt{'(8 9 10 11 12)})$

```
;*lemmas for b1b

;;;If the k's entry in lp is between 9 and 12,
;;;then the k's entry in gp is iether 3 or 4 by lg.
```

THEOREM: lp9-12-then-k-in-g34
$((k \in \text{nset}(n)) \land \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)}) \land \text{lg}(n, lp, gp))$
$\rightarrow \quad \text{union-at-n}(gp, k, \texttt{'(3 4)})$

```
;;;If the k's entry in lp is between 8 and 12, and
;;;it is not 8, then it is certainly between 9 and 12.
```

THEOREM: un8-12-then-l8-or-l9-12
$(\text{union-at-n}(lp, k, \texttt{'(8 9 10 11 12)}) \land (\neg \text{at}(lp, k, 8)))$
$\rightarrow \quad \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)})$

```
;;;;;;;;;;;;;;;;;;;;;;;;;; defn.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;;Well-formed-state.
```

DEFINITION:
$\text{ws}(n, l, g)$
$= \quad ((n \in \mathbf{N})$
$\land \quad \text{listp}(l)$
$\land \quad \text{listp}(g)$
$\land \quad (\text{length}(l) = n)$
$\land \quad (\text{length}(g) = n)$
$\land \quad \text{all-union}(l, n, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$
$\land \quad \text{all-union}(g, n, \texttt{'(0 1 2 3 4)}))$

EVENT: Disable ws.

```
;;;Transitions.
```

DEFINITION:
$\text{rhoi0}(n, i, l, g, lp, gp)$
$= \quad (\text{at}(l, i, \texttt{0}) \land (gp = g) \land (lp = \text{move}(l, i, \texttt{1})))$

Definition:
rhoi1a $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{1}) \land (gp = g) \land (lp = \text{move}\,(l,\ i,\ \texttt{2})))$

Definition:
rhoi1b $(n,\ i,\ l,\ g,\ lp,\ gp) = (\text{at}\,(l,\ i,\ \texttt{1}) \land (g = gp) \land (lp = l))$

Definition:
rhoi2 $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{2}) \land (lp = \text{move}\,(l,\ i,\ \texttt{3})) \land (gp = \text{move}\,(g,\ i,\ \texttt{1})))$

Definition:
rhoi3a $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{3})$
$\quad \land \quad (gp = g)$
$\quad \land \quad (lp = \text{move}\,(l,\ i,\ \texttt{4}))$
$\quad \land \quad (\neg\ \text{exist-union}\,(g,\ n,\ \texttt{'(3 4)})))$

Definition:
rhoi3b $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{3})$
$\quad \land \quad (gp = g)$
$\quad \land \quad (lp = l)$
$\quad \land \quad \text{exist-union}\,(g,\ n,\ \texttt{'(3 4)}))$

Definition:
rhoi4 $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{4}) \land (gp = \text{move}\,(g,\ i,\ \texttt{3})) \land (lp = \text{move}\,(l,\ i,\ \texttt{5})))$

Definition:
rhoi5a $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{5})$
$\quad \land \quad (gp = g)$
$\quad \land \quad \text{exist-union}\,(g,\ n,\ \texttt{'(1)})$
$\quad \land \quad (lp = \text{move}\,(l,\ i,\ \texttt{6})))$

Definition:
rhoi5b $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{5})$
$\quad \land \quad (gp = g)$
$\quad \land \quad (\neg\ \text{exist-union}\,(g,\ n,\ \texttt{'(1)}))$
$\quad \land \quad (lp = \text{move}\,(l,\ i,\ \texttt{8})))$

Definition:
rhoi6 $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{6}) \land (gp = \text{move}\,(g,\ i,\ \texttt{2})) \land (lp = \text{move}\,(l,\ i,\ \texttt{7})))$

Definition:
rhoi7a $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{7})$
$\quad \land \quad \text{exist-union}\,(g,\ n,\ \texttt{'(4)})$
$\quad \land \quad (lp = \text{move}\,(l,\ i,\ \texttt{8}))$
$\quad \land \quad (gp = g))$

Definition:
rhoi7b $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{7})$
$\quad \land \quad (\neg\ \text{exist-union}\,(g,\ n,\ \texttt{'(4)}))$
$\quad \land \quad (lp = l)$
$\quad \land \quad (gp = g))$

Definition:
rhoi8 $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{8}) \land (gp = \text{move}\,(g,\ i,\ \texttt{4})) \land (lp = \text{move}\,(l,\ i,\ \texttt{9})))$

Definition:
phi9 $(i,\ n,\ g)$
$= \quad$ **if** $(g \simeq \textbf{nil}) \lor (i \notin \mathbf{N}) \lor (n \notin \mathbf{N})$ **then f**
elseif $n = \texttt{0}$ **then t**
else $((n \not< i) \land \text{phi9}\,(i,\ n-1,\ g))$
$\quad \lor \quad (\text{union-at-n}\,(g,\ n,\ \texttt{'(0 1)}) \land \text{phi9}\,(i,\ n-1,\ g))$ **endif**

Event: Disable phi9.

Definition:
rhoi9a $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{9}) \land (gp = g) \land \text{phi9}\,(i,\ n,\ g) \land (lp = \text{move}\,(l,\ i,\ \texttt{10})))$

Definition:
rhoi9b $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{9}) \land (gp = g) \land (\neg\ \text{phi9}\,(i,\ n,\ g)) \land (lp = l))$

Definition:
rhoi10 $(n,\ i,\ l,\ g,\ lp,\ gp)$
$= \quad (\text{at}\,(l,\ i,\ \texttt{10}) \land (lp = \text{move}\,(l,\ i,\ \texttt{11})) \land (gp = g))$

Definition:
phi11 $(i,\ n,\ g)$
$= \quad$ **if** $(g \simeq \textbf{nil}) \lor (i \notin \mathbf{N}) \lor (n \notin \mathbf{N})$ **then f**
elseif $n = \texttt{0}$ **then t**
else $((i \not< n) \land \text{phi11}\,(i,\ n-1,\ g))$
$\quad \lor \quad ((\neg\ \text{union-at-n}\,(g,\ n,\ \texttt{'(2 3)}))$
$\quad\quad \land \quad \text{phi11}\,(i,\ n-1,\ g))$ **endif**

EVENT: Disable phi11.

DEFINITION:
rhoi11a (n, i, l, g, lp, gp)
$= \quad (\text{at}(l, i, \texttt{11})$
$\quad \wedge \quad (gp = g)$
$\quad \wedge \quad \text{phi11}(i, n, g)$
$\quad \wedge \quad (lp = \text{move}(l, i, \texttt{12})))$

DEFINITION:
rhoi11b (n, i, l, g, lp, gp)
$= \quad (\text{at}(l, i, \texttt{11}) \wedge (gp = g) \wedge (\neg\ \text{phi11}(i, n, g)) \wedge (lp = l))$

DEFINITION:
rhoi12 (n, i, l, g, lp, gp)
$= \quad (\text{at}(l, i, \texttt{12}) \wedge (gp = \text{move}(g, i, \texttt{0})) \wedge (lp = \text{move}(l, i, \texttt{0})))$

```
;;;The transition operates on i'th.
```

DEFINITION:
rhoi (n, i, l, g, lp, gp)
$= \quad (\text{rhoi0}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi1a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi1b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi2}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi3a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi3b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi4}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi5a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi5b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi6}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi7a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi7b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi8}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi9a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi9b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi10}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi11a}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi11b}(n, i, l, g, lp, gp)$
$\quad \vee \quad \text{rhoi12}(n, i, l, g, lp, gp))$

EVENT: Disable rhoi.

```
;;;  Propositions
```

```
;;;a0
```

DEFINITION:
$\mathrm{a0}\,(n,\ l,\ k)$
$= \quad (((k \in \mathrm{nset}\,(n)) \wedge \text{exist-union}\,(l,\ n,\ \texttt{'(8 9 10 11 12)}))$
$\quad \rightarrow \quad (\neg\ \mathrm{at}\,(l,\ k,\ \texttt{4})))$

EVENT: Disable a0.

```
;;;a1
```

DEFINITION:
$\mathrm{a1}\,(n,\ l,\ g)$
$= \quad (\text{exist-union}\,(l,\ n,\ \texttt{'(8 9 10 11 12)})$
$\quad \rightarrow \quad \text{exist-intersect-8-12-3-4}\,(n,\ l,\ g))$

EVENT: Disable a1.

```
;;;; a2
```

DEFINITION:
a2-at-n1-n2 (*n1*, *n2*, *l*)
= **if** union-at-n (*l*, *n1*, `'(10 11 12)`)
then ¬ union-at-n (*l*, *n2*, `'(5 6 7 8 9 10 11 12)`)
else t endif

EVENT: Disable a2-at-n1-n2.

DEFINITION:
a2-at-n2 (*n1*, *n2*, *l*)
= **if** $n2 \simeq \texttt{0}$ **then t**
elseif $n2 \not< n1$ **then** a2-at-n2 (*n1*, $n2 - 1$, *l*)
else a2-at-n1-n2 (*n1*, *n2*, *l*) ∧ a2-at-n2 (*n1*, $n2 - 1$, *l*) **endif**

EVENT: Disable a2-at-n2.

DEFINITION:
a2 (*n1*, *n2*, *l*)
= **if** $n1 \simeq \texttt{0}$ **then t**
else a2-at-n2 (*n1*, *n2*, *l*) ∧ a2 ($n1 - 1$, *n2*, *l*) **endif**

Event: Disable a2.

```
;;;a3
```

Definition:
a3-at-n1-n2 ($n1$, $n2$, l, g)
= **if** at (l, $n1$, `12`) ∧ union-at-n (l, $n2$, `'(5 6 7 8 9 10 11 12)`)
then at (g, $n2$, `4`)
else t endif

Event: Disable a3-at-n1-n2.

Definition:
a3-at-n2 ($n1$, $n2$, l, g)
= **if** $n2 \simeq$ `0` **then t**
else a3-at-n1-n2 ($n1$, $n2$, l, g) ∧ a3-at-n2 ($n1$, $n2 - 1$, l, g) **endif**

Event: Disable a3-at-n2.

Definition:
a3 ($n1$, $n2$, l, g)
= **if** $n1 \simeq$ `0` **then t**
else a3-at-n2 ($n1$, $n2$, l, g) ∧ a3 ($n1 - 1$, $n2$, l, g) **endif**

Event: Disable a3.

```
;;;;;;;;;;;;;;;;;;;;;;;;;; basic.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;;ws implies that n is a number.
```

Theorem: ws-num-n
ws (n, l, g) → ($n \in \mathbf{N}$)

```
;;;ws implies that l is a list.
```

Theorem: ws-list-l
ws (n, l, g) → listp (l)

```
;;;ws implies that g is a list.
```

Theorem: ws-list-g
ws (n, l, g) → listp (g)

```
;;;ws implies that length of l is n.
```

Theorem: ws-ln-l
$\text{ws}(n, l, g) \rightarrow (\text{length}(l) = n)$

```
;;;ws implies that length of g is n.
```

Theorem: ws-ln-g
$\text{ws}(n, l, g) \rightarrow (\text{length}(g) = n)$

```
;;;ws and rho imply that lp is a list.
```

Theorem: ws-ln-lp
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp)) \rightarrow \text{listp}(lp)$

```
;;;ws and rho imply that gp is a list.
```

Theorem: ws-ln-gp
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp)) \rightarrow \text{listp}(gp)$

```
;;;ws implies that n is nonzero.
```

Theorem: ws-n-not-0
$\text{ws}(n, l, g) \rightarrow (n \not\simeq 0)$

Theorem: n-not-0
$\text{ws}(n, l, g) \rightarrow (n \in \text{nset}(n))$

```
;*the rho! lemmas

;;;Auxiliary lemma.
```

Theorem: lm-l-rholemma
$$
\begin{array}{ll}
(\text{listp}(l) & \\
\land & (j \in \text{nset}(\text{length}(l))) \\
\land & (k \in \text{nset}(\text{length}(l))) \\
\land & \text{rhoi}(n, k, l, g, lp, gp) \\
\land & (k \neq j)) \\
\rightarrow & (\text{nth}(l, j) = \text{nth}(lp, j))
\end{array}
$$

Event: Disable lm-l-rholemma.

```
;;;Rholemma for list l.
```

Theorem: l-rholemma
$$
\begin{array}{ll}
(\text{ws}(n, l, g) & \\
\land & (j \in \text{nset}(n)) \\
\land & (k \in \text{nset}(n)) \\
\land & \text{rhoi}(n, k, l, g, lp, gp) \\
\land & (k \neq j)) \\
\rightarrow & (\text{nth}(l, j) = \text{nth}(lp, j))
\end{array}
$$

```
;;;Auxiliary lemma.
```

THEOREM: lm-g-rholemma
(listp (g)
 ∧ $(j \in \text{nset}(\text{length}(g)))$
 ∧ $(k \in \text{nset}(\text{length}(g)))$
 ∧ $\text{rhoi}(n, k, l, g, lp, gp)$
 ∧ $(k \neq j))$
→ $(\text{nth}(g, j) = \text{nth}(gp, j))$

EVENT: Disable lm-g-rholemma.

```
;;;Rholemma for list g.
```

THEOREM: g-rholemma
$(\text{ws}(n, l, g)$
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ $\text{rhoi}(n, k, l, g, lp, gp)$
 ∧ $(k \neq j))$
→ $(\text{nth}(g, j) = \text{nth}(gp, j))$

```
;;; lp-gp-same-l-g

;;;Another version of Rholemma for l.
;;;It applies to (union-at-n l j m) in stead of
;;;(nth l j).
```

THEOREM: lp-same-l
$(\text{ws}(n, l, g)$
 ∧ $\text{listp}(m)$
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ $\text{rhoi}(n, k, l, g, lp, gp)$
 ∧ $(j \neq k)$
 ∧ $\text{union-at-n}(l, j, m))$
→ $\text{union-at-n}(lp, j, m)$

```
;;;Contrast to the one above,
;;;the order of l and lp is reversed.
```

THEOREM: l-same-lp
$(\text{ws}(n, l, g)$
 ∧ $\text{listp}(m)$
 ∧ $(j \in \text{nset}(n))$

$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $(j \neq k)$
$\wedge$ $\text{union-at-n}(lp, j, m))$
$\rightarrow$ $\text{union-at-n}(l, j, m)$

THEOREM: lp-same-l-not
$(\text{ws}(n, l, g)$
$\wedge$ $\text{listp}(m)$
$\wedge$ $(j \in \text{nset}(n))$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $(j \neq k)$
$\wedge$ $(\neg\, \text{union-at-n}(l, j, m)))$
$\rightarrow$ $(\neg\, \text{union-at-n}(lp, j, m))$

```
;;;Another version of Rholemma for g.
```

THEOREM: gp-same-g
$(\text{ws}(n, l, g)$
$\wedge$ $\text{listp}(m)$
$\wedge$ $(j \in \text{nset}(n))$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $(j \neq k)$
$\wedge$ $\text{union-at-n}(g, j, m))$
$\rightarrow$ $\text{union-at-n}(gp, j, m)$

```
;;;Contrast to the one above,
;;;the order of g and gp is reversed.
```

THEOREM: g-same-gp
$(\text{ws}(n, l, g)$
$\wedge$ $\text{listp}(m)$
$\wedge$ $(j \in \text{nset}(n))$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $(j \neq k)$
$\wedge$ $\text{union-at-n}(gp, j, m))$
$\rightarrow$ $\text{union-at-n}(g, j, m)$

```
;;;It applies to (at l j m) in stead of
;;;(nth l j).
```

THEOREM: l-same-lp-at

$(\text{ws}(n, l, g)$
$\wedge \quad (j \in \text{nset}(n))$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad (m \in \mathbf{N})$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (j \neq k)$
$\wedge \quad \text{at}(lp, j, m))$
$\rightarrow \quad \text{at}(l, j, m)$

Theorem: gp-same-g-at
$(\text{ws}(n, l, g)$
$\wedge \quad (j \in \text{nset}(n))$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad (m \in \mathbf{N})$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (j \neq k)$
$\wedge \quad \text{at}(g, j, m))$
$\rightarrow \quad \text{at}(gp, j, m)$

Theorem: l-same-lp-at-not
$(\text{ws}(n, l, g)$
$\wedge \quad (m \in \mathbf{N})$
$\wedge \quad (j \in \text{nset}(n))$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (j \neq k)$
$\wedge \quad (\neg\ \text{at}(l, j, m)))$
$\rightarrow \quad (\neg\ \text{at}(lp, j, m))$

```
;*basic properties of a2

;;;Auxiliary lemma.
```

Theorem: lm-a2-at-n2-a2-at-n1-n2
$((n \in \mathbf{N}) \wedge (k \in \mathbf{N}) \wedge (j \in \text{nset}(n)) \wedge (j < k) \wedge \text{a2-at-n2}(k, n, l))$
$\rightarrow \quad \text{a2-at-n1-n2}(k, j, l)$

Event: Disable lm-a2-at-n2-a2-at-n1-n2.

Theorem: a2-at-n2-a2-at-n1-n2
$(\text{ws}(n, l, g)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad (j \in \text{nset}(n))$
$\wedge \quad (j < k)$
$\wedge \quad \text{a2-at-n2}(k, n, l))$
$\rightarrow \quad \text{a2-at-n1-n2}(k, j, l)$

THEOREM: lm-a2-a2-at-n2
$((n \in \mathbf{N}) \land (i \in \mathbf{N}) \land (k \in \text{nset}(n)) \land \text{a2}(n, i, l)) \rightarrow \text{a2-at-n2}(k, i, l)$

THEOREM: a2-a2-at-n2
$(\text{ws}(n, l, g) \land (i \in \text{nset}(n)) \land (k \in \text{nset}(n)) \land \text{a2}(n, i, l))$
$\rightarrow \quad \text{a2-at-n2}(k, i, l)$

```
;*basic properties of a3
```

THEOREM: lm-a3-at-n2-a3-at-n1-n2
$((n \in \mathbf{N}) \land (u \in \mathbf{N}) \land (j \in \text{nset}(n)) \land \text{a3-at-n2}(u, n, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(u, j, l, g)$

EVENT: Disable lm-a3-at-n2-a3-at-n1-n2.

THEOREM: a3-at-n2-a3-at-n1-n2
$(\text{ws}(n, l, g) \land (u \in \text{nset}(n)) \land (j \in \text{nset}(n)) \land \text{a3-at-n2}(u, n, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(u, j, l, g)$

THEOREM: lm-a3-a3-at-n2
$((n \in \mathbf{N}) \land (i \in \mathbf{N}) \land (u \in \text{nset}(n)) \land \text{a3}(n, i, l, g))$
$\rightarrow \quad \text{a3-at-n2}(u, i, l, g)$

EVENT: Disable lm-a3-a3-at-n2.

THEOREM: a3-a3-at-n2
$(\text{ws}(n, l, g) \land (i \in \text{nset}(n)) \land (u \in \text{nset}(n)) \land \text{a3}(n, i, l, g))$
$\rightarrow \quad \text{a3-at-n2}(u, i, l, g)$

```
;;;;;;;Instances used in the proofs of
;;;;;;;a1 a2 and a3.

;;;(a2 n n l) and (a3 n n l g) are involved
;;;in double bounded quantifiers
;;;\forall i \leq n \forall j \leq n,
;;;with their quantifier-free formulas
;;;(a3-at-n1-n2 i j l g) and (a2-at-n1-n2 i j l)
;;;respectively. What follows are all instances of
;;;the following type: If (a3 n n l g) holds, then
;;;in particular, so its instance
;;;(a3-at-n1-n2 i j l g) does.

;;;The instances are i and j.
```

THEOREM: a3-i-j-a3-at-n1-n2
$(\text{ws}(n, l, g) \land (i \in \text{nset}(n)) \land (j \in \text{nset}(n)) \land \text{a3}(n, n, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(i, j, l, g)$

```
;;;The instances are k and
;;;(exist-union lp n '(8 9 10 11 12)).
```

THEOREM: a3-ex-a3-at-n1-n2
$(\text{ws}(n, l, g)$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{a3}(n, n, l, g)$
$\land \quad \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)}))$
$\rightarrow \quad \text{a3-at-n1-n2}(k, \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)}), l, g)$

THEOREM: a2-n-a2-at-n2
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{a2}(n, n, l)) \rightarrow \text{a2-at-n2}(k, n, l)$

```
;;;The instances are i and j.
```

THEOREM: a2-i-j-a2-at-n1-n2
$(\text{ws}(n, l, g)$
$\land \quad (i \in \text{nset}(n))$
$\land \quad (j \in \text{nset}(n))$
$\land \quad \text{a2}(n, n, l)$
$\land \quad (j < i))$
$\rightarrow \quad \text{a2-at-n1-n2}(i, j, l)$

```
;;;;;;;;;;;;;;;;;;;;;;;;;;;;; ws.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
```

THEOREM: j-eq-k-move-member-g
$(\text{listp}(g) \land \text{listp}(m) \land (i \in m) \land (k \in \text{nset}(\text{length}(g))))$
$\rightarrow \quad (\text{nth}(\text{move}(g, k, i), k) \in m)$

THEOREM: j-neq-k-move-member-g
$(\text{listp}(g)$
$\land \quad \text{listp}(m)$
$\land \quad (k \in \text{nset}(\text{length}(g)))$
$\land \quad (j \neq k)$
$\land \quad (i \in m)$
$\land \quad (\text{nth}(g, j) \in m))$
$\rightarrow \quad (\text{nth}(\text{move}(g, k, i), j) \in m)$

THEOREM: move-member-g
$(\text{listp}(g)$
$\land \quad \text{listp}(m)$
$\land \quad (i \in m)$

$\wedge \quad (k \in \text{nset}(\text{length}(g)))$
$\wedge \quad (\text{nth}(g, j) \in m))$
$\rightarrow \quad (\text{nth}(\text{move}(g, k, i), j) \in m)$

THEOREM: move-member-l
$(\text{listp}(l)$
$\wedge \quad \text{listp}(m)$
$\wedge \quad (j \in \mathbf{N})$
$\wedge \quad (i \in m)$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (\text{nth}(l, j) \in m))$
$\rightarrow \quad (\text{nth}(\text{move}(l, k, i), j) \in m)$

THEOREM: ws-union-g
$\text{ws}(n, l, g) \rightarrow \text{all-union}(g, n,$ `'(0 1 2 3 4)`$)$

THEOREM: ws-union-l
$\text{ws}(n, l, g) \rightarrow \text{all-union}(l, n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: rho0-preserves-union-g
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi0}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho1a-preserves-union-g
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi1a}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho1b-preserves-union-g
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi1b}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: lm-rho2-preserves-union-g
$(\text{listp}(g)$
$\wedge \quad (\text{length}(g) = n)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{all-union}(g, j,$ `'(0 1 2 3 4)`$)$
$\wedge \quad \text{rhoi2}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, j,$ `'(0 1 2 3 4)`$)$

THEOREM: rho2-preserves-union-g
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi2}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho3a-preserves-union-g
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi3a}(n, k, l, g, lp, gp))$
$\rightarrow \quad \text{all-union}(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho3b-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi3b}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: lm-rho4-preserves-union-g
$(\text{listp}(g)$
$\land$ $(\text{length}(g) = n)$
$\land$ $(k \in \text{nset}(n))$
$\land$ all-union $(g, j,$ `'(0 1 2 3 4)`$)$
$\land$ $\text{rhoi4}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, j,$ `'(0 1 2 3 4)`$)$

THEOREM: rho4-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi4}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho5a-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi5a}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho5b-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi5b}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: lm-rho6-preserves-union-g
$(\text{listp}(g)$
$\land$ $(\text{length}(g) = n)$
$\land$ $(k \in \text{nset}(n))$
$\land$ all-union $(g, j,$ `'(0 1 2 3 4)`$)$
$\land$ $\text{rhoi6}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, j,$ `'(0 1 2 3 4)`$)$

THEOREM: rho6-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi6}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho7a-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi7a}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: rho7b-preserves-union-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi7b}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(gp, n,$ `'(0 1 2 3 4)`$)$

THEOREM: lm-rho8-preserves-union-g
(listp (g)
 ∧ (length $(g) = n$)
 ∧ $(k \in$ nset $(n))$
 ∧ all-union $(g,\ j,$ `'(0 1 2 3 4)`)
 ∧ rhoi8 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ j,$ `'(0 1 2 3 4)`)

THEOREM: rho8-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi8 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: rho9a-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi9a $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: rho9b-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi9b $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: lm-rho10-preserves-union-g
(listp (g)
 ∧ (length $(g) = n$)
 ∧ $(k \in$ nset $(n))$
 ∧ all-union $(g,\ j,$ `'(0 1 2 3 4)`)
 ∧ rhoi10 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ j,$ `'(0 1 2 3 4)`)

THEOREM: rho10-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi10 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: rho11a-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi11a $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: rho11b-preserves-union-g
(ws $(n,\ l,\ g) \land (k \in$ nset $(n)) \land$ rhoi11b $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`)

THEOREM: lm-rho12-preserves-union-g
(listp (g)
 ∧ (length $(g) = n$)
 ∧ $(k \in$ nset $(n))$
 ∧ all-union $(g,\ j,$ `'(0 1 2 3 4)`)
 ∧ rhoi12 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ j,$ `'(0 1 2 3 4)`)

Theorem: rho12-preserves-union-g
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi12 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`$)$

Theorem: rho-preserves-union-g
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(gp,\ n,$ `'(0 1 2 3 4)`$)$

Theorem: lm-rho0-preserves-union-l
(listp (l)
∧ (length $(l) = n$)
∧ $(k \in$ nset $(n))$
∧ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
∧ rhoi0 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: rho0-preserves-union-l
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi0 $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: lm-rho1a-preserves-union-l
(listp (l)
∧ (length $(l) = n$)
∧ $(k \in$ nset $(n))$
∧ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
∧ rhoi1a $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: rho1a-preserves-union-l
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi1a $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: lm-rho1b-preserves-union-l
(listp (l)
∧ (length $(l) = n$)
∧ $(k \in$ nset $(n))$
∧ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
∧ rhoi1b $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: rho1b-preserves-union-l
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi1b $(n,\ k,\ l,\ g,\ lp,\ gp))$
→ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

Theorem: lm-rho2-preserves-union-l

$(\text{listp}(l)$
$\wedge$ $(\text{length}(l) = n)$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
$\wedge$ rhoi2 $(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: rho2-preserves-union-l
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi2}(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: lm-rho3a-preserves-union-l
$(\text{listp}(l)$
$\wedge$ $(\text{length}(l) = n)$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
$\wedge$ rhoi3a $(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: rho3a-preserves-union-l
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi3a}(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: lm-rho3b-preserves-union-l
$(\text{listp}(l)$
$\wedge$ $(\text{length}(l) = n)$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
$\wedge$ rhoi3b $(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: rho3b-preserves-union-l
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi3b}(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: lm-rho4-preserves-union-l
$(\text{listp}(l)$
$\wedge$ $(\text{length}(l) = n)$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ all-union $(l,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
$\wedge$ rhoi4 $(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ j,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: rho4-preserves-union-l
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi4}(n,\ k,\ l,\ g,\ lp,\ gp))$
$\rightarrow$ all-union $(lp,\ n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: lm-rho5a-preserves-union-l
(listp (l)
 ∧ (length (l) = n)
 ∧ ($k \in$ nset (n))
 ∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
 ∧ rhoi5a (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho5a-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi5a (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho5b-preserves-union-l
(listp (l)
 ∧ (length (l) = n)
 ∧ ($k \in$ nset (n))
 ∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
 ∧ rhoi5b (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho5b-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi5b (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho6-preserves-union-l
(listp (l)
 ∧ (length (l) = n)
 ∧ ($k \in$ nset (n))
 ∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
 ∧ rhoi6 (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho6-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi6 (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho7a-preserves-union-l
(listp (l)
 ∧ (length (l) = n)
 ∧ ($k \in$ nset (n))
 ∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
 ∧ rhoi7a (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho7a-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi7a (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho7b-preserves-union-l
$$(\text{listp}(l) \land (\text{length}(l) = n) \land (k \in \text{nset}(n)) \land \text{all-union}(l, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)}) \land \text{rhoi7b}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: rho7b-preserves-union-l
$$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi7b}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, n, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: lm-rho8-preserves-union-l
$$(\text{listp}(l) \land (\text{length}(l) = n) \land (k \in \text{nset}(n)) \land \text{all-union}(l, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)}) \land \text{rhoi8}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: rho8-preserves-union-l
$$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi8}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, n, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: lm-rho9a-preserves-union-l
$$(\text{listp}(l) \land (\text{length}(l) = n) \land (k \in \text{nset}(n)) \land \text{all-union}(l, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)}) \land \text{rhoi9a}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: rho9a-preserves-union-l
$$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi9a}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, n, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: lm-rho9b-preserves-union-l
$$(\text{listp}(l) \land (\text{length}(l) = n) \land (k \in \text{nset}(n)) \land \text{all-union}(l, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)}) \land \text{rhoi9b}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, j, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: rho9b-preserves-union-l
$$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi9b}(n, k, l, g, lp, gp)) \rightarrow \text{all-union}(lp, n, \texttt{'(0 1 2 3 4 5 6 7 8 9 10 11 12)})$$

THEOREM: lm-rho10-preserves-union-l
(listp (l)
∧ (length (l) = n)
∧ ($k \in$ nset (n))
∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
∧ rhoi10 (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho10-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi10 (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho11a-preserves-union-l
(listp (l)
∧ (length (l) = n)
∧ ($k \in$ nset (n))
∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
∧ rhoi11a (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho11a-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi11a (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho11b-preserves-union-l
(listp (l)
∧ (length (l) = n)
∧ ($k \in$ nset (n))
∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
∧ rhoi11b (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho11b-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi11b (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: lm-rho12-preserves-union-l
(listp (l)
∧ (length (l) = n)
∧ ($k \in$ nset (n))
∧ all-union (l, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)
∧ rhoi12 (n, k, l, g, lp, gp))
→ all-union (lp, j, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho12-preserves-union-l
(ws (n, l, g) ∧ ($k \in$ nset (n)) ∧ rhoi12 (n, k, l, g, lp, gp))
→ all-union (lp, n, `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`)

THEOREM: rho-preserves-union-l
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp))$
$\rightarrow$ all-union $(lp, n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$

THEOREM: lm-rho-preserves-ln-l
$(\text{listp}(l) \land (\text{length}(l) = n) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp))$
$\rightarrow$ $(\text{length}(lp) = n)$

THEOREM: rho-preserves-ln-l
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp))$
$\rightarrow$ $(\text{length}(lp) = n)$

THEOREM: lm-rho-preserves-ln-g
$(\text{listp}(g) \land (\text{length}(g) = n) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp))$
$\rightarrow$ $(\text{length}(gp) = n)$

THEOREM: rho-preserves-ln-g
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp))$
$\rightarrow$ $(\text{length}(gp) = n)$

THEOREM: lm-rho-preserves-ws
$((n \in \mathbf{N})$
$\land$ listp (lp)
$\land$ listp (gp)
$\land$ $(\text{length}(lp) = n)$
$\land$ $(\text{length}(gp) = n)$
$\land$ all-union $(lp, n,$ `'(0 1 2 3 4 5 6 7 8 9 10 11 12)`$)$
$\land$ all-union $(gp, n,$ `'(0 1 2 3 4)`$))$
$\rightarrow$ ws (n, lp, gp)

THEOREM: rho-preserves-ws
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp)) \rightarrow \text{ws}(n, lp, gp)$

```
;;;;;;;;;;;;;;;;;;;;;;;;;;; lg.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;;rhoi0
```

THEOREM: n-neq-k-rhoi0
$(\text{listp}(l)$
$\land$ listp (g)
$\land$ $(n \in \mathbf{N})$
$\land$ $(k \in \text{nset}(\text{length}(l)))$
$\land$ $(k \neq n)$
$\land$ at $(l, k,$ `0`$)$
$\land$ lg-at-n $(n, l, g))$
$\rightarrow$ lg-at-n $(n, \text{move}(l, k,$ `1`$), g)$

EVENT: Disable n-neq-k-rhoi0.

THEOREM: n-eq-k-rhoi0
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ at (l, k, `0`)
 $\wedge$ lg-at-n (k, l, g))
$\rightarrow$ lg-at-n (k, move (l, k, `1`), g)

EVENT: Disable n-eq-k-rhoi0.

THEOREM: lg-at-rhoi0
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($n \in \mathbf{N}$)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ at (l, k, `0`)
 $\wedge$ lg-at-n (n, l, g))
$\rightarrow$ lg-at-n (n, move (l, k, `1`), g)

EVENT: Disable lg-at-rhoi0.

THEOREM: lg-rhoi0
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ ($n \in \mathbf{N}$)
 $\wedge$ at (l, k, `0`)
 $\wedge$ lg (n, l, g))
$\rightarrow$ lg (n, move (l, k, `1`), g)

EVENT: Disable lg-rhoi0.

THEOREM: rhoi0-preserves-lg
(ws (n, l, g) $\wedge$ ($k \in$ nset (n)) $\wedge$ rhoi0 (n, k, l, g, lp, gp) $\wedge$ lg (n, l, g))
$\rightarrow$ lg (n, lp, gp)

`;;;rhoi1a`

THEOREM: n-neq-k-rhoi1a
(listp (l)
 $\wedge$ listp (g)

$\wedge$ $(n \in \mathbf{N})$
$\wedge$ $(k \in \text{nset}(\text{length}(l)))$
$\wedge$ $(k \neq n)$
$\wedge$ $\text{at}(l, k, \mathtt{1})$
$\wedge$ $\text{lg-at-n}(n, l, g))$
$\rightarrow$ $\text{lg-at-n}(n, \text{move}(l, k, \mathtt{2}), g)$

EVENT: Disable n-neq-k-rhoi1a.

THEOREM: n-eq-k-rhoi1a
$(\text{listp}(l)$
$\wedge$ $\text{listp}(g)$
$\wedge$ $(k \in \text{nset}(\text{length}(l)))$
$\wedge$ $\text{at}(l, k, \mathtt{1})$
$\wedge$ $\text{lg-at-n}(k, l, g))$
$\rightarrow$ $\text{lg-at-n}(n, \text{move}(l, k, \mathtt{2}), g)$

EVENT: Disable n-eq-k-rhoi1a.

THEOREM: lg-at-rhoi1a
$(\text{listp}(l)$
$\wedge$ $\text{listp}(g)$
$\wedge$ $(n \in \mathbf{N})$
$\wedge$ $(k \in \text{nset}(\text{length}(l)))$
$\wedge$ $\text{at}(l, k, \mathtt{1})$
$\wedge$ $\text{lg-at-n}(n, l, g))$
$\rightarrow$ $\text{lg-at-n}(n, \text{move}(l, k, \mathtt{2}), g)$

EVENT: Disable lg-at-rhoi1a.

THEOREM: lg-rhoi1a
$(\text{listp}(l)$
$\wedge$ $\text{listp}(g)$
$\wedge$ $(k \in \text{nset}(\text{length}(l)))$
$\wedge$ $(n \in \mathbf{N})$
$\wedge$ $\text{at}(l, k, \mathtt{1})$
$\wedge$ $\text{lg}(n, l, g))$
$\rightarrow$ $\text{lg}(n, \text{move}(l, k, \mathtt{2}), g)$

EVENT: Disable lg-rhoi1a.

THEOREM: rhoi1a-preserves-lg
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi1a}(n, k, l, g, lp, gp) \wedge \text{lg}(n, l, g))$
$\rightarrow$ $\text{lg}(n, lp, gp)$

`;;;rhoi1b`

Theorem: rhoi1b-preserves-lg
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi1b}(n, k, l, g, lp, gp) \wedge \text{lg}(n, l, g))$
$\rightarrow \quad \text{lg}(n, lp, gp)$

`;;;rhoi2`

Theorem: n-neq-k-rhoi2
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (k \neq n)$
$\wedge \quad \text{at}(l, k, 2)$
$\wedge \quad \text{lg-at-n}(n, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \text{move}(l, k, 3), \text{move}(g, k, 1))$

Event: Disable n-neq-k-rhoi2.

Theorem: n-eq-k-rhoi2
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l, k, 2)$
$\wedge \quad \text{lg-at-n}(k, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \text{move}(l, k, 3), \text{move}(g, k, 1))$

Event: Disable n-eq-k-rhoi2.

Theorem: lg-at-rhoi2
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l, k, 2)$
$\wedge \quad \text{lg-at-n}(n, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \text{move}(l, k, 3), \text{move}(g, k, 1))$

Event: Disable lg-at-rhoi2.

Theorem: lg-rhoi2
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$

$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad \text{at}(l,\ k,\ \mathtt{2})$
$\wedge \quad \text{lg}(n,\ l,\ g))$
$\rightarrow \quad \text{lg}(n,\ \text{move}(l,\ k,\ \mathtt{3}),\ \text{move}(g,\ k,\ \mathtt{1}))$

Event: Disable lg-rhoi2.

Theorem: rhoi2-preserves-lg
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi2}(n,\ k,\ l,\ g,\ lp,\ gp) \wedge \text{lg}(n,\ l,\ g))$
$\rightarrow \quad \text{lg}(n,\ lp,\ gp)$

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;;;rhoi3a
```

Theorem: n-neq-k-rhoi3a
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (k \neq n)$
$\wedge \quad \text{at}(l,\ k,\ \mathtt{3})$
$\wedge \quad \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \mathtt{4}),\ g)$

Event: Disable n-neq-k-rhoi3a.

Theorem: n-eq-k-rhoi3a
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l,\ k,\ \mathtt{3})$
$\wedge \quad \text{lg-at-n}(k,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(k,\ \text{move}(l,\ k,\ \mathtt{4}),\ g)$

Event: Disable n-eq-k-rhoi3a.

Theorem: lg-at-rhoi3a
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l,\ k,\ \mathtt{3})$
$\wedge \quad \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \mathtt{4}),\ g)$

EVENT: Disable lg-at-rhoi3a.

THEOREM: lg-rhoi3a
(listp (l)
 $\land$ listp (g)
 $\land$ ($k \in$ nset (length (l)))
 $\land$ ($n \in \mathbf{N}$)
 $\land$ at (l, k, 3)
 $\land$ lg (n, l, g))
$\rightarrow$ lg (n, move (l, k, 4), g)

EVENT: Disable lg-rhoi3a.

THEOREM: rhoi3a-preserves-lg
(ws (n, l, g) $\land$ ($k \in$ nset (n)) $\land$ rhoi3a (n, k, l, g, lp, gp) $\land$ lg (n, l, g))
$\rightarrow$ lg (n, lp, gp)

`;;;rhoi3b`

THEOREM: rhoi3b-preserves-lg
(ws (n, l, g) $\land$ ($k \in$ nset (n)) $\land$ rhoi3b (n, k, l, g, lp, gp) $\land$ lg (n, l, g))
$\rightarrow$ lg (n, lp, gp)

`;;;rhoi4`

THEOREM: n-neq-k-rhoi4
(listp (l)
 $\land$ listp (g)
 $\land$ ($n \in \mathbf{N}$)
 $\land$ ($k \in$ nset (length (l)))
 $\land$ ($k \neq n$)
 $\land$ at (l, k, 4)
 $\land$ lg-at-n (n, l, g))
$\rightarrow$ lg-at-n (n, move (l, k, 5), move (g, k, 3))

EVENT: Disable n-neq-k-rhoi4.

THEOREM: n-eq-k-rhoi4
(listp (l)
 $\land$ listp (g)
 $\land$ ($k \in$ nset (length (l)))
 $\land$ at (l, k, 4)
 $\land$ lg-at-n (k, l, g))
$\rightarrow$ lg-at-n (n, move (l, k, 5), move (g, k, 3))

EVENT: Disable n-eq-k-rhoi4.

THEOREM: lg-at-rhoi4
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($n \in \mathbf{N}$)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ at (l, k, `4`)
 $\wedge$ lg-at-n (n, l, g))
 $\rightarrow$ lg-at-n (n, move (l, k, `5`), move (g, k, `3`))

EVENT: Disable lg-at-rhoi4.

THEOREM: lg-rhoi4
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ ($n \in \mathbf{N}$)
 $\wedge$ at (l, k, `4`)
 $\wedge$ lg (n, l, g))
 $\rightarrow$ lg (n, move (l, k, `5`), move (g, k, `3`))

EVENT: Disable lg-rhoi4.

THEOREM: rhoi4-preserves-lg
(ws (n, l, g) $\wedge$ ($k \in$ nset (n)) $\wedge$ rhoi4 (n, k, l, g, lp, gp) $\wedge$ lg (n, l, g))
 $\rightarrow$ lg (n, lp, gp)

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;;;rhoi5a
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THEOREM: n-neq-k-rhoi5a
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ ($n \in \mathbf{N}$)
 $\wedge$ ($k \in$ nset (length (l)))
 $\wedge$ ($k \neq n$)
 $\wedge$ at (l, k, `5`)
 $\wedge$ lg-at-n (n, l, g))
 $\rightarrow$ lg-at-n (n, move (l, k, `6`), g)

EVENT: Disable n-neq-k-rhoi5a.

THEOREM: n-eq-k-rhoi5a

$(\text{listp}\,(l)$
$\wedge\quad \text{listp}\,(g)$
$\wedge\quad (k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge\quad \text{at}\,(l,\ k,\ \texttt{5})$
$\wedge\quad \text{lg-at-n}\,(k,\ l,\ g))$
$\rightarrow\quad \text{lg-at-n}\,(k,\ \text{move}\,(l,\ k,\ \texttt{6}),\ g)$

EVENT: Disable n-eq-k-rhoi5a.

THEOREM: lg-at-rhoi5a
$(\text{listp}\,(l)$
$\wedge\quad \text{listp}\,(g)$
$\wedge\quad (n \in \mathbf{N})$
$\wedge\quad (k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge\quad \text{at}\,(l,\ k,\ \texttt{5})$
$\wedge\quad \text{lg-at-n}\,(n,\ l,\ g))$
$\rightarrow\quad \text{lg-at-n}\,(n,\ \text{move}\,(l,\ k,\ \texttt{6}),\ g)$

EVENT: Disable lg-at-rhoi5a.

THEOREM: lg-rhoi5a
$(\text{listp}\,(l)$
$\wedge\quad \text{listp}\,(g)$
$\wedge\quad (k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge\quad (n \in \mathbf{N})$
$\wedge\quad \text{at}\,(l,\ k,\ \texttt{5})$
$\wedge\quad \text{lg}\,(n,\ l,\ g))$
$\rightarrow\quad \text{lg}\,(n,\ \text{move}\,(l,\ k,\ \texttt{6}),\ g)$

EVENT: Disable lg-rhoi5a.

THEOREM: rhoi5a-preserves-lg
$(\text{ws}\,(n,\ l,\ g) \wedge (k \in \text{nset}\,(n)) \wedge \text{rhoi5a}\,(n,\ k,\ l,\ g,\ lp,\ gp) \wedge \text{lg}\,(n,\ l,\ g))$
$\rightarrow\quad \text{lg}\,(n,\ lp,\ gp)$

`;;;rhoi5b`

THEOREM: n-neq-k-rhoi5b
$(\text{listp}\,(l)$
$\wedge\quad \text{listp}\,(g)$
$\wedge\quad (n \in \mathbf{N})$
$\wedge\quad (k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge\quad (k \neq n)$

$\wedge$ at $(l,\ k,\ \texttt{5})$
$\wedge$ lg-at-n $(n,\ l,\ g))$
$\rightarrow$ lg-at-n $(n,\ \text{move}\,(l,\ k,\ \texttt{8}),\ g)$

Event: Disable n-neq-k-rhoi5b.

Theorem: n-eq-k-rhoi5b
(listp (l)
$\wedge$ listp (g)
$\wedge$ $(k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge$ at $(l,\ k,\ \texttt{5})$
$\wedge$ lg-at-n $(k,\ l,\ g))$
$\rightarrow$ lg-at-n $(k,\ \text{move}\,(l,\ k,\ \texttt{8}),\ g)$

Event: Disable n-eq-k-rhoi5b.

Theorem: lg-at-rhoi5b
(listp (l)
$\wedge$ listp (g)
$\wedge$ $(n \in \mathbf{N})$
$\wedge$ $(k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge$ at $(l,\ k,\ \texttt{5})$
$\wedge$ lg-at-n $(n,\ l,\ g))$
$\rightarrow$ lg-at-n $(n,\ \text{move}\,(l,\ k,\ \texttt{8}),\ g)$

Event: Disable lg-at-rhoi5b.

Theorem: lg-rhoi5b
(listp (l)
$\wedge$ listp (g)
$\wedge$ $(k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge$ $(n \in \mathbf{N})$
$\wedge$ at $(l,\ k,\ \texttt{5})$
$\wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n,\ \text{move}\,(l,\ k,\ \texttt{8}),\ g)$

Event: Disable lg-rhoi5b.

Theorem: rhoi5b-preserves-lg
(ws $(n,\ l,\ g) \wedge (k \in \text{nset}\,(n)) \wedge$ rhoi5b $(n,\ k,\ l,\ g,\ lp,\ gp) \wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n,\ lp,\ gp)$

`;;;rhoi6`

THEOREM: n-neq-k-rhoi6
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ $(n \in \mathbf{N})$
 $\wedge$ $(k \in$ nset (length $(l)))$
 $\wedge$ $(k \neq n)$
 $\wedge$ at $(l,\ k,\ 6)$
 $\wedge$ lg-at-n $(n,\ l,\ g))$
$\rightarrow$ lg-at-n $(n$, move $(l,\ k,\ 7)$, move $(g,\ k,\ 2))$

EVENT: Disable n-neq-k-rhoi6.

THEOREM: n-eq-k-rhoi6
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ $(k \in$ nset (length $(l)))$
 $\wedge$ at $(l,\ k,\ 6)$
 $\wedge$ lg-at-n $(k,\ l,\ g))$
$\rightarrow$ lg-at-n $(n$, move $(l,\ k,\ 7)$, move $(g,\ k,\ 2))$

EVENT: Disable n-eq-k-rhoi6.

THEOREM: lg-at-rhoi6
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ $(n \in \mathbf{N})$
 $\wedge$ $(k \in$ nset (length $(l)))$
 $\wedge$ at $(l,\ k,\ 6)$
 $\wedge$ lg-at-n $(n,\ l,\ g))$
$\rightarrow$ lg-at-n $(n$, move $(l,\ k,\ 7)$, move $(g,\ k,\ 2))$

EVENT: Disable lg-at-rhoi6.

THEOREM: lg-rhoi6
(listp (l)
 $\wedge$ listp (g)
 $\wedge$ $(k \in$ nset (length $(l)))$
 $\wedge$ $(n \in \mathbf{N})$
 $\wedge$ at $(l,\ k,\ 6)$
 $\wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n$, move $(l,\ k,\ 7)$, move $(g,\ k,\ 2))$

EVENT: Disable lg-rhoi6.

Theorem: rhoi6-preserves-lg
$(\text{ws}(n,\ l,\ g) \land (k \in \text{nset}(n)) \land \text{rhoi6}(n,\ k,\ l,\ g,\ lp,\ gp) \land \text{lg}(n,\ l,\ g))$
$\rightarrow\ \ \text{lg}(n,\ lp,\ gp)$

`;;;rhoi7a`

Theorem: n-neq-k-rhoi7a
$(\text{listp}(l)$
$\land\ \ \text{listp}(g)$
$\land\ \ (n \in \mathbf{N})$
$\land\ \ (k \in \text{nset}(\text{length}(l)))$
$\land\ \ (k \neq n)$
$\land\ \ \text{at}(l,\ k,\ \mathtt{7})$
$\land\ \ \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow\ \ \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \mathtt{8}),\ g)$

Event: Disable n-neq-k-rhoi7a.

Theorem: n-eq-k-rhoi7a
$(\text{listp}(l)$
$\land\ \ \text{listp}(g)$
$\land\ \ (k \in \text{nset}(\text{length}(l)))$
$\land\ \ \text{at}(l,\ k,\ \mathtt{7})$
$\land\ \ \text{lg-at-n}(k,\ l,\ g))$
$\rightarrow\ \ \text{lg-at-n}(k,\ \text{move}(l,\ k,\ \mathtt{8}),\ g)$

Event: Disable n-eq-k-rhoi7a.

Theorem: lg-at-rhoi7a
$(\text{listp}(l)$
$\land\ \ \text{listp}(g)$
$\land\ \ (n \in \mathbf{N})$
$\land\ \ (k \in \text{nset}(\text{length}(l)))$
$\land\ \ \text{at}(l,\ k,\ \mathtt{7})$
$\land\ \ \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow\ \ \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \mathtt{8}),\ g)$

Event: Disable lg-at-rhoi7a.

Theorem: lg-rhoi7a
$(\text{listp}(l)$
$\land\ \ \text{listp}(g)$
$\land\ \ (k \in \text{nset}(\text{length}(l)))$

$\wedge$ $(n \in \mathbf{N})$
$\wedge$ at $(l,\ k,\ \mathtt{7})$
$\wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n,\ \text{move}\,(l,\ k,\ \mathtt{8}),\ g)$

EVENT: Disable lg-rhoi7a.

THEOREM: rhoi7a-preserves-lg
(ws $(n,\ l,\ g)$ $\wedge$ $(k \in \text{nset}\,(n))$ $\wedge$ rhoi7a $(n,\ k,\ l,\ g,\ lp,\ gp)$ $\wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n,\ lp,\ gp)$

`;;;rhoi7b`

THEOREM: rhoi7b-preserves-lg
(ws $(n,\ l,\ g)$ $\wedge$ $(k \in \text{nset}\,(n))$ $\wedge$ rhoi7b $(n,\ k,\ l,\ g,\ lp,\ gp)$ $\wedge$ lg $(n,\ l,\ g))$
$\rightarrow$ lg $(n,\ lp,\ gp)$

`;;;rhoi8`

THEOREM: n-neq-k-rhoi8
(listp (l)
$\wedge$ listp (g)
$\wedge$ $(n \in \mathbf{N})$
$\wedge$ $(k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge$ $(k \neq n)$
$\wedge$ at $(l,\ k,\ \mathtt{8})$
$\wedge$ lg-at-n $(n,\ l,\ g))$
$\rightarrow$ lg-at-n $(n,\ \text{move}\,(l,\ k,\ \mathtt{9}),\ \text{move}\,(g,\ k,\ \mathtt{4}))$

EVENT: Disable n-neq-k-rhoi8.

THEOREM: n-eq-k-rhoi8
(listp (l)
$\wedge$ listp (g)
$\wedge$ $(k \in \text{nset}\,(\text{length}\,(l)))$
$\wedge$ at $(l,\ k,\ \mathtt{8})$
$\wedge$ lg-at-n $(k,\ l,\ g))$
$\rightarrow$ lg-at-n $(n,\ \text{move}\,(l,\ k,\ \mathtt{9}),\ \text{move}\,(g,\ k,\ \mathtt{4}))$

EVENT: Disable n-eq-k-rhoi8.

THEOREM: lg-at-rhoi8
(listp (l)
$\wedge$ listp (g)

$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l,\ k,\ \texttt{8})$
$\wedge \quad \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \texttt{9}),\ \text{move}(g,\ k,\ \texttt{4}))$

EVENT: Disable lg-at-rhoi8.

THEOREM: lg-rhoi8
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad \text{at}(l,\ k,\ \texttt{8})$
$\wedge \quad \text{lg}(n,\ l,\ g))$
$\rightarrow \quad \text{lg}(n,\ \text{move}(l,\ k,\ \texttt{9}),\ \text{move}(g,\ k,\ \texttt{4}))$

EVENT: Disable lg-rhoi8.

THEOREM: rhoi8-preserves-lg
$(\text{ws}(n,\ l,\ g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi8}(n,\ k,\ l,\ g,\ lp,\ gp) \wedge \text{lg}(n,\ l,\ g))$
$\rightarrow \quad \text{lg}(n,\ lp,\ gp)$

`;;;rhoi9a`

THEOREM: n-neq-k-rhoi9a
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (n \in \mathbf{N})$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad (k \neq n)$
$\wedge \quad \text{at}(l,\ k,\ \texttt{9})$
$\wedge \quad \text{lg-at-n}(n,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(n,\ \text{move}(l,\ k,\ \texttt{10}),\ g)$

EVENT: Disable n-neq-k-rhoi9a.

THEOREM: n-eq-k-rhoi9a
$(\text{listp}(l)$
$\wedge \quad \text{listp}(g)$
$\wedge \quad (k \in \text{nset}(\text{length}(l)))$
$\wedge \quad \text{at}(l,\ k,\ \texttt{9})$
$\wedge \quad \text{lg-at-n}(k,\ l,\ g))$
$\rightarrow \quad \text{lg-at-n}(k,\ \text{move}(l,\ k,\ \texttt{10}),\ g)$

EVENT: Disable n-eq-k-rhoi9a.

THEOREM: lg-at-rhoi9a
$(\text{listp}(l)$
$\land\ \text{listp}(g)$
$\land\ (n \in \mathbf{N})$
$\land\ (k \in \text{nset}(\text{length}(l)))$
$\land\ \text{at}(l, k, \texttt{9})$
$\land\ \text{lg-at-n}(n, l, g))$
$\to\ \text{lg-at-n}(n, \text{move}(l, k, \texttt{10}), g)$

EVENT: Disable lg-at-rhoi9a.

THEOREM: lg-rhoi9a
$(\text{listp}(l)$
$\land\ \text{listp}(g)$
$\land\ (k \in \text{nset}(\text{length}(l)))$
$\land\ (n \in \mathbf{N})$
$\land\ \text{at}(l, k, \texttt{9})$
$\land\ \text{lg}(n, l, g))$
$\to\ \text{lg}(n, \text{move}(l, k, \texttt{10}), g)$

EVENT: Disable lg-rhoi9a.

THEOREM: rhoi9a-preserves-lg
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi9a}(n, k, l, g, lp, gp) \land \text{lg}(n, l, g))$
$\to\ \text{lg}(n, lp, gp)$

`;;;rhoi9b`

THEOREM: rhoi9b-preserves-lg
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi9b}(n, k, l, g, lp, gp) \land \text{lg}(n, l, g))$
$\to\ \text{lg}(n, lp, gp)$

`;;;rhoi10`

THEOREM: n-neq-k-rhoi10
$(\text{listp}(l)$
$\land\ \text{listp}(g)$
$\land\ (n \in \mathbf{N})$
$\land\ (k \in \text{nset}(\text{length}(l)))$
$\land\ (k \neq n)$
$\land\ \text{at}(l, k, \texttt{10})$
$\land\ \text{lg-at-n}(n, l, g))$
$\to\ \text{lg-at-n}(n, \text{move}(l, k, \texttt{11}), g)$

Event: Disable n-neq-k-rhoi10.

Theorem: n-eq-k-rhoi10
(listp (l)
 ∧ listp (g)
 ∧ $(k \in$ nset (length $(l)))$
 ∧ at $(l,\ k,$ `10`$)$
 ∧ lg-at-n $(k,\ l,\ g))$
→ lg-at-n $(k,$ move $(l,\ k,$ `11`$),\ g)$

Event: Disable n-eq-k-rhoi10.

Theorem: lg-at-rhoi10
(listp (l)
 ∧ listp (g)
 ∧ $(n \in \mathbf{N})$
 ∧ $(k \in$ nset (length $(l)))$
 ∧ at $(l,\ k,$ `10`$)$
 ∧ lg-at-n $(n,\ l,\ g))$
→ lg-at-n $(n,$ move $(l,\ k,$ `11`$),\ g)$

Event: Disable lg-at-rhoi10.

Theorem: lg-rhoi10
(listp (l)
 ∧ listp (g)
 ∧ $(k \in$ nset (length $(l)))$
 ∧ $(n \in \mathbf{N})$
 ∧ at $(l,\ k,$ `10`$)$
 ∧ lg $(n,\ l,\ g))$
→ lg $(n,$ move $(l,\ k,$ `11`$),\ g)$

Event: Disable lg-rhoi10.

Theorem: rhoi10-preserves-lg
(ws $(n,\ l,\ g)$ ∧ $(k \in$ nset $(n))$ ∧ rhoi10 $(n,\ k,\ l,\ g,\ lp,\ gp)$ ∧ lg $(n,\ l,\ g))$
→ lg $(n,\ lp,\ gp)$

`;;;rhoi11a`

Theorem: n-neq-k-rhoi11a
(listp (l)
 ∧ listp (g)

$\land$ $(n \in \mathbf{N})$
$\land$ $(k \in \text{nset}(\text{length}(l)))$
$\land$ $(k \neq n)$
$\land$ $\text{at}(l,\ k,\ \texttt{11})$
$\land$ $\text{lg-at-n}(n,\ l,\ g))$
$\rightarrow$ $\text{lg-at-n}(n,\ \text{move}(l,\ k,\ \texttt{12}),\ g)$

EVENT: Disable n-neq-k-rhoi11a.

THEOREM: n-eq-k-rhoi11a
$(\text{listp}(l)$
$\land$ $\text{listp}(g)$
$\land$ $(k \in \text{nset}(\text{length}(l)))$
$\land$ $\text{at}(l,\ k,\ \texttt{11})$
$\land$ $\text{lg-at-n}(k,\ l,\ g))$
$\rightarrow$ $\text{lg-at-n}(k,\ \text{move}(l,\ k,\ \texttt{12}),\ g)$

EVENT: Disable n-eq-k-rhoi11a.

THEOREM: lg-at-rhoi11a
$(\text{listp}(l)$
$\land$ $\text{listp}(g)$
$\land$ $(n \in \mathbf{N})$
$\land$ $(k \in \text{nset}(\text{length}(l)))$
$\land$ $\text{at}(l,\ k,\ \texttt{11})$
$\land$ $\text{lg-at-n}(n,\ l,\ g))$
$\rightarrow$ $\text{lg-at-n}(n,\ \text{move}(l,\ k,\ \texttt{12}),\ g)$

EVENT: Disable lg-at-rhoi11a.

THEOREM: lg-rhoi11a
$(\text{listp}(l)$
$\land$ $\text{listp}(g)$
$\land$ $(k \in \text{nset}(\text{length}(l)))$
$\land$ $(n \in \mathbf{N})$
$\land$ $\text{at}(l,\ k,\ \texttt{11})$
$\land$ $\text{lg}(n,\ l,\ g))$
$\rightarrow$ $\text{lg}(n,\ \text{move}(l,\ k,\ \texttt{12}),\ g)$

EVENT: Disable lg-rhoi11a.

THEOREM: rhoi11a-preserves-lg
$(\text{ws}(n,\ l,\ g) \land (k \in \text{nset}(n)) \land \text{rhoi11a}(n,\ k,\ l,\ g,\ lp,\ gp) \land \text{lg}(n,\ l,\ g))$
$\rightarrow$ $\text{lg}(n,\ lp,\ gp)$

`;;;rhoi11b`

THEOREM: rhoi11b-preserves-lg
$(\mathrm{ws}(n, l, g) \land (k \in \mathrm{nset}(n)) \land \mathrm{rhoi11b}(n, k, l, g, lp, gp) \land \mathrm{lg}(n, l, g))$
$\rightarrow \quad \mathrm{lg}(n, lp, gp)$

`;;;rhoi12`

THEOREM: n-neq-k-rhoi12
$(\mathrm{listp}(l)$
$\land \quad \mathrm{listp}(g)$
$\land \quad (n \in \mathbf{N})$
$\land \quad (k \in \mathrm{nset}(\mathrm{length}(l)))$
$\land \quad (k \neq n)$
$\land \quad \mathrm{at}(l, k, \mathtt{12})$
$\land \quad \text{lg-at-n}(n, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \mathrm{move}(l, k, \mathtt{0}), \mathrm{move}(g, k, \mathtt{0}))$

EVENT: Disable n-neq-k-rhoi12.

THEOREM: n-eq-k-rhoi12
$(\mathrm{listp}(l)$
$\land \quad \mathrm{listp}(g)$
$\land \quad (k \in \mathrm{nset}(\mathrm{length}(l)))$
$\land \quad \mathrm{at}(l, k, \mathtt{12})$
$\land \quad \text{lg-at-n}(k, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \mathrm{move}(l, k, \mathtt{0}), \mathrm{move}(g, k, \mathtt{0}))$

EVENT: Disable n-eq-k-rhoi12.

THEOREM: lg-at-rhoi12
$(\mathrm{listp}(l)$
$\land \quad \mathrm{listp}(g)$
$\land \quad (n \in \mathbf{N})$
$\land \quad (k \in \mathrm{nset}(\mathrm{length}(l)))$
$\land \quad \mathrm{at}(l, k, \mathtt{12})$
$\land \quad \text{lg-at-n}(n, l, g))$
$\rightarrow \quad \text{lg-at-n}(n, \mathrm{move}(l, k, \mathtt{0}), \mathrm{move}(g, k, \mathtt{0}))$

EVENT: Disable lg-at-rhoi12.

THEOREM: lg-rhoi12
$(\mathrm{listp}(l)$
$\land \quad \mathrm{listp}(g)$

$\land \quad (k \in \text{nset}(\text{length}(l)))$
$\land \quad (n \in \mathbf{N})$
$\land \quad \text{at}(l, k, \texttt{12})$
$\land \quad \text{lg}(n, l, g))$
$\rightarrow \quad \text{lg}(n, \text{move}(l, k, \texttt{0}), \text{move}(g, k, \texttt{0}))$

EVENT: Disable lg-rhoi12.

THEOREM: rhoi12-preserves-lg
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi12}(n, k, l, g, lp, gp) \land \text{lg}(n, l, g))$
$\rightarrow \quad \text{lg}(n, lp, gp)$

THEOREM: rho-preserves-lg
$(\text{ws}(n, l, g) \land (k \in \text{nset}(n)) \land \text{rhoi}(n, k, l, g, lp, gp) \land \text{lg}(n, l, g))$
$\rightarrow \quad \text{lg}(n, lp, gp)$

EVENT: Disable rhoi0-preserves-lg.

EVENT: Disable rhoi1a-preserves-lg.

EVENT: Disable rhoi1b-preserves-lg.

EVENT: Disable rhoi2-preserves-lg.

EVENT: Disable rhoi3a-preserves-lg.

EVENT: Disable rhoi3b-preserves-lg.

EVENT: Disable rhoi4-preserves-lg.

EVENT: Disable rhoi5a-preserves-lg.

EVENT: Disable rhoi5b-preserves-lg.

EVENT: Disable rhoi6-preserves-lg.

EVENT: Disable rhoi7a-preserves-lg.

EVENT: Disable rhoi7b-preserves-lg.

EVENT: Disable rhoi8-preserves-lg.

EVENT: Disable rhoi9a-preserves-lg.

EVENT: Disable rhoi9b-preserves-lg.

EVENT: Disable rhoi10-preserves-lg.

EVENT: Disable rhoi11a-preserves-lg.

EVENT: Disable rhoi11b-preserves-lg.

EVENT: Disable rhoi12-preserves-lg.

```
;;;;;;;;;;;;;;;;;;;;;;;;;;; a0.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;;(exist-union lp n '(8 9 10 11 12))  and
;;;(not (exist-union l n '(8 9 10 11 12)))
;;;implies that k is the witness of
;;;(exist-union lp n '(8 9 10 11 12)). This
;;;proposition would have been more natural
;;;if we had been able to prove:
;;;(prove-lemma exist-l8-12 (rewrite)
;;;   (implies (and (ws n l g)
;;;                 (member k (nset n))
;;;                 (rhoi n k l g lp gp)
;;;                 (exist-union lp n '(8 9 10 11 12))
;;;                 (not (exist-union l n '(8 9 10 11 12))))
;;;            (equal k (exist-union lp n '(8 9 10 11 12))))).
;;;However Bmp refused to rewrite equal clause.
```

THEOREM: exist-l8-12
(ws (n, l, g)
 $\wedge$ $(k \in \text{nset}(n))$
 $\wedge$ rhoi (n, k, l, g, lp, gp)
 $\wedge$ exist-union $(lp, n,$ '(8 9 10 11 12))
 $\wedge$ $(k \neq$ exist-union $(lp, n,$ '(8 9 10 11 12))))
$\rightarrow$ exist-union $(l, n,$ '(8 9 10 11 12))

```
;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(not (exist-union l n '(8 9 10 11 12))) hold,
```

;;;then the k's entry of lp is between 8..12.

THEOREM: k-in-lp8-12
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ exist-union (lp, n, '(8 9 10 11 12))
 ∧ (¬ exist-union (l, n, '(8 9 10 11 12))))
→ union-at-n (lp, k, '(8 9 10 11 12))

;;;If k's entry in lp is between 8..12 and
;;;k's entry of l is not between 8..12,
;;;then k's entry of l is either 5 or 7.

THEOREM: k-not-in-l8-12-then-l57
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ union-at-n (lp, k, '(8 9 10 11 12))
 ∧ (¬ union-at-n (l, k, '(8 9 10 11 12)))
 ∧ (¬ at (l, k, 7)))
→ at (l, k, 5)

;;;If k's entry in lp is between 8..12 and
;;;(not (exist-union l n '(8 9 10 11 12))) holds,
;;;then k's entry of l is either 5 or 7.

THEOREM: k-in-l57
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ union-at-n (lp, k, '(8 9 10 11 12))
 ∧ (¬ exist-union (l, n, '(8 9 10 11 12)))
 ∧ (¬ at (l, k, 7)))
→ at (l, k, 5)

;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(not (exist-union l n '(8 9 10 11 12))) hold,
;;;then the k's entry of l is between either 5 or 7.

THEOREM: ex-k-in-l57
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ exist-union (lp, n, '(8 9 10 11 12))

$\wedge$ ($\neg$ exist-union (l, n, `'(8 9 10 11 12)`))
$\wedge$ ($\neg$ at (l, k, 7)))
$\rightarrow$ at (l, k, 5)

```
;;;Auxiliary lemma for ex-cond-rhoi5.
```

THEOREM: cond-rhoi5
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ union-at-n (lp, k, `'(8 9 10 11 12)`)
$\wedge$ at (l, k, 5))
$\rightarrow$ ($\neg$ exist-union (g, n, `'(1)`))

```
;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(not (exist-union l n '(8 9 10 11 12))) and
;;;the k's entry in l is 5 then
;;;(not (exist-union g n '(1))) holds.
```

THEOREM: ex-cond-rhoi5
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ exist-union (lp, n, `'(8 9 10 11 12)`)
$\wedge$ ($\neg$ exist-union (l, n, `'(8 9 10 11 12)`))
$\wedge$ at (l, k, 5))
$\rightarrow$ ($\neg$ exist-union (g, n, `'(1)`))

```
;;;Auxiliary lemma for ex-cond-rhoi7.
```

THEOREM: cond-rhoi7
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ union-at-n (lp, k, `'(8 9 10 11 12)`)
$\wedge$ at (l, k, 7))
$\rightarrow$ exist-union (g, n, `'(4)`)

```
;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(not (exist-union l n '(8 9 10 11 12))) and
;;;the k's entry in l is 7, then
;;;(exist-union g n '(4)) holds.
```

THEOREM: ex-cond-rhoi7
(ws (n, l, g)

$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)})$
$\wedge \quad (\neg\, \text{exist-union}(l, n, \texttt{'(8 9 10 11 12)}))$
$\wedge \quad \text{at}(l, k, 7))$
$\rightarrow \quad \text{exist-union}(g, n, \texttt{'(4)})$

```
;;;If (exist-union lp n '(8 9 10 11 12))
;;;and (not (exist-union l n '(8 9 10 11 12))),
;;;then (not (exist-union g n '(1))) holds.
```

THEOREM: l5-only-lp8
$(\text{ws}(n, l, g)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (\neg\, \text{exist-union}(l, n, \texttt{'(8 9 10 11 12)}))$
$\wedge \quad \text{lg}(n, l, g)$
$\wedge \quad \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)}))$
$\rightarrow \quad (\neg\, \text{exist-union}(g, n, \texttt{'(1)}))$

```
;;;If j is not equal to k and j's entry of l
;;;is neither 3 or 4, then j's entry of lp
;;;is not 4.
```

THEOREM: j-neq-k-j-not-in-lp4
$(\text{ws}(n, l, g)$
$\wedge \quad (j \in \text{nset}(n))$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (j \neq k)$
$\wedge \quad (\neg\, \text{union-at-n}(l, j, \texttt{'(3 4)})))$
$\rightarrow \quad (\neg\, \text{at}(lp, j, 4))$

```
;;;If k's entry of l is neither 3 or 4, then
;;;k's entry of lp is not 4.
```

THEOREM: j-eq-k-j-not-in-lp4
$(\text{ws}(n, l, g)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (\neg\, \text{union-at-n}(l, k, \texttt{'(3 4)})))$
$\rightarrow \quad (\neg\, \text{at}(lp, k, 4))$

```
;;;If j's entry of l is neither 3 or 4, then
;;;j's entry of lp is not 4.
```

Theorem: lp4-empty

$$
\begin{array}{ll}
& (\mathrm{ws}\,(n,\ l,\ g) \\
\wedge & (j \in \mathrm{nset}\,(n)) \\
\wedge & (k \in \mathrm{nset}\,(n)) \\
\wedge & \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp) \\
\wedge & (\neg\ \text{union-at-n}\,(l,\ j,\ \texttt{'(3 4)}))) \\
\rightarrow & (\neg\ \mathrm{at}\,(lp,\ j,\ 4))
\end{array}
$$

```
;;;If (not (exist-union l n '(8 9 10 11 12)))
;;; and (exist-union lp n '(8 9 10 11 12)) hold,
;;;then there is no entry 4 in l.
```

Theorem: l8-l12-empty

$$
\begin{array}{ll}
& (\mathrm{ws}\,(n,\ l,\ g) \\
\wedge & (j \in \mathrm{nset}\,(n)) \\
\wedge & (k \in \mathrm{nset}\,(n)) \\
\wedge & \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp) \\
\wedge & (\neg\ \text{exist-union}\,(l,\ n,\ \texttt{'(8 9 10 11 12)})) \\
\wedge & \mathrm{lg}\,(n,\ l,\ g)) \\
\rightarrow & \mathrm{a0}\,(n,\ lp,\ j)
\end{array}
$$

```
;;;If (exist-union g n '(3 4)) holds and
;;;the k's entry in l is not 4, then
;;;the k's entry in lp is not 4 either.
;;;(Doorway is locked.)
```

Theorem: dwy-lckd

$$
\begin{array}{ll}
& (\mathrm{ws}\,(n,\ l,\ g) \\
\wedge & (k \in \mathrm{nset}\,(n)) \\
\wedge & \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp) \\
\wedge & \text{exist-union}\,(g,\ n,\ \texttt{'(3 4)}) \\
\wedge & (\neg\ \mathrm{at}\,(l,\ k,\ 4))) \\
\rightarrow & (\neg\ \mathrm{at}\,(lp,\ k,\ 4))
\end{array}
$$

```
;;;If (exist-union l n '(8 9 10 11 12))
;;;holds and j is equal to k, then
;;;j's entry in lp is not 4.
```

Theorem: j-eq-k-l8-l12-nonemp

$$
\begin{array}{ll}
& (\mathrm{ws}\,(n,\ l,\ g) \\
\wedge & (k \in \mathrm{nset}\,(n)) \\
\wedge & \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp) \\
\wedge & \text{exist-union}\,(l,\ n,\ \texttt{'(8 9 10 11 12)}) \\
\wedge & \mathrm{a0}\,(n,\ l,\ k) \\
\wedge & \mathrm{a1}\,(n,\ l,\ g)) \\
\rightarrow & \mathrm{a0}\,(n,\ lp,\ k)
\end{array}
$$

```
;;;If (exist-union l n '(8 9 10 11 12))
;;; holds and j is not equal to k, then
;;;the j's entry in lp is not 4.
```

THEOREM: j-neq-k-l8-l12-nonemp
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}\,(n))$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(j \neq k)$
 ∧ a0 $(n,\ l,\ j)$
 ∧ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$))$
→ a0 $(n,\ lp,\ j)$

```
;;;If (exist-union l n '(8 9 10 11 12))
;;;holds then there is no entry 4 in lp.
;;;The order of the use hints is critical.
;;;Change the order and we fail.
```

THEOREM: l8-l12-nonemp
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}\,(n))$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$)$
 ∧ a0 $(n,\ l,\ j)$
 ∧ a1 $(n,\ l,\ g))$
→ a0 $(n,\ lp,\ j)$

```
;;;If (exist-union lp n '(8 9 10 11 12))
;;;holds, then there is no entry 4 in lp.
```

THEOREM: rho-preserves-a0
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}\,(n))$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a0 $(n,\ l,\ j)$
 ∧ a1 $(n,\ l,\ g))$
→ a0 $(n,\ lp,\ j)$

```
;;;;;;;;;;;;;;;;;;;;;;;;;;;; a1.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;* ep-l8-12

;;;Auxiliary lemma for at-gp-rhoi5
```

THEOREM: gp-rhoi5
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ union-at-n (lp, k, `'(8 9 10 11 12)`)
 ∧ at (l, k, `5`)
 ∧ at (g, k, `3`))
→ at (gp, k, `3`)

```
;;;If (not (exist-union l n '(8 9 10 11 12))),
;;;(exist-union lp n '(8 9 10 11 12)) and the k's
;;;entry in l is 5 then the k's entry in gp is 3.
```

THEOREM: ex-gp-rhoi5
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ lg (n, l, g)
 ∧ (¬ exist-union (l, n, `'(8 9 10 11 12)`))
 ∧ exist-union (lp, n, `'(8 9 10 11 12)`)
 ∧ at (l, k, `5`))
→ at (gp, k, `3`)

```
;;;If (not (exist-union l n '(8 9 10 11 12)))
;;;and (exist-union lp n '(8 9 10 11 12)) holds,
;;;then the k's entry is either 3 or 4.
```

THEOREM: k-in-gp34
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ lg (n, l, g)
 ∧ (¬ exist-union (l, n, `'(8 9 10 11 12)`))
 ∧ exist-union (lp, n, `'(8 9 10 11 12)`))
→ union-at-n (gp, k, `'(3 4)`)

```
;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(not (exist-union l n '(8 9 10 11 12))) holds,
;;;then so does (exist-intersect-8-12-3-4 n lp gp).
```

THEOREM: lm-a1-ep-l8-12
(ws (n, l, g)
 ∧ ($k \in$ nset (n))
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ exist-union (lp, n, `'(8 9 10 11 12)`)

$\wedge$ (¬ exist-union (l, n, `'(8 9 10 11 12)`))
$\wedge$ lg (n, l, g))
$\rightarrow$ exist-intersect-8-12-3-4 (n, lp, gp)

```
;;;If (not (exist-union l n '(8 9 10 11 12))) holds,
;;;then so does a1.
```

THEOREM: a1-ep-l8-12
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ lg (n, l, g)
$\wedge$ (¬ exist-union (l, n, `'(8 9 10 11 12)`)))
$\rightarrow$ a1 (n, lp, gp)

```
;* nep-l8-12

;;;If (exist-intersect-8-12-3-4 n l g) holds and
;;;k is not its witness then
;;;(exist-intersect-8-12-3-4 n lp gp) holds.
```

THEOREM: int-k-not-ex-int
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ ($k \neq$ exist-intersect-8-12-3-4 (n, l, g))
$\wedge$ exist-intersect-8-12-3-4 (n, l, g))
$\rightarrow$ exist-intersect-8-12-3-4 (n, lp, gp)

```
;;;If (exist-union l n '(8 9 10 11 12)) holds and
;;;k's enrty is not between 8 and 12 then
;;;(exist-intersect-8-12-3-4 n lp gp) holds.
;;; j \neq k
```

THEOREM: a1-k-not-in-l8-12-nep-l8-12
(ws (n, l, g)
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ exist-union (l, n, `'(8 9 10 11 12)`)
$\wedge$ (¬ union-at-n (l, k, `'(8 9 10 11 12)`))
$\wedge$ a1 (n, l, g))
$\rightarrow$ exist-intersect-8-12-3-4 (n, lp, gp)

```
;* k-in-l8-11
;;;If the k's entry in l is between 8 and 11,
;;;then the k's entry in lp is between 9 and 12.
;;;We need rho-preserves-lg.
```

THEOREM: l8-11-k-in-lp9-12
(ws $(n,\ l,\ g)$
 $\wedge$ $(k \in \text{nset}\,(n))$
 $\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\wedge$ union-at-n $(l,\ k,$ '(8 9 10 11)))
 $\rightarrow$ union-at-n $(lp,\ k,$ '(9 10 11 12))

```
;;;If the k's entry in l is between 8 and 11,
;;;then the k's entry in lp is between 8 and 12
;;;and the entry in gp is either 3 or 4.
;;;l8-11-k-in-lp9-12, un9-12-then-un8-12 and
;;;rho-preserves-lg are used.
```

THEOREM: lm-a1-k-in-l8-11-nep-l8-12
(ws $(n,\ l,\ g)$
 $\wedge$ $(k \in \text{nset}\,(n))$
 $\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\wedge$ union-at-n $(l,\ k,$ '(8 9 10 11))
 $\wedge$ lg $(n,\ l,\ g)$)
 $\rightarrow$ (union-at-n $(lp,\ k,$ '(8 9 10 11 12)) $\wedge$ union-at-n $(gp,\ k,$ '(3 4)))

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds,
;;;and the k's entry in l is between 8 and 11 then
;;;(exist-intersect-8-12-3-4 n lp gp) holds.
;;; j \eq k and k \in l8-11
```

THEOREM: a1-k-in-l8-11-nep-l8-12
(ws $(n,\ l,\ g)$
 $\wedge$ $(k \in \text{nset}\,(n))$
 $\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\wedge$ lg $(n,\ l,\ g)$
 $\wedge$ exist-union $(lp,\ n,$ '(8 9 10 11 12))
 $\wedge$ union-at-n $(l,\ k,$ '(8 9 10 11)))
 $\rightarrow$ exist-intersect-8-12-3-4 $(n,\ lp,\ gp)$

```
;;;If the k's entry in l is 12 then the k's entry in l is 0.
```

THEOREM: k-in-lp0
(ws $(n,\ l,\ g)$ $\wedge$ $(k \in \text{nset}\,(n))$ $\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$ $\wedge$ at $(l,\ k,$ 12))
 $\rightarrow$ at $(lp,\ k,$ 0)

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds
;;;and k's entry in l is 12, then k is not the
;;;witness of (exist-union lp n '(8 9 10 11 12)).
```

THEOREM: k-not-ex-lp8-12
$(\text{ws}(n, l, g)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)})$
$\wedge \quad \text{at}(l, k, \texttt{12}))$
$\rightarrow \quad (k \neq \text{exist-union}(lp, n, \texttt{'(8 9 10 11 12)}))$

```
;;;If the k's entry in lp is 8,
;;;then k's entry in l is either 5 or 7.
```

THEOREM: lp8-k-in-l57
$(\text{ws}(n, l, g) \wedge (k \in \text{nset}(n)) \wedge \text{rhoi}(n, k, l, g, lp, gp) \wedge \text{at}(lp, k, \texttt{8}))$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(5 7)})$

```
;;;If the k's entry in lp is 8,
;;;then k's entry in l is between 5 and 12.
```

THEOREM: k-in-lp8-then-l5-12
$(\text{ws}(n, l, g) \wedge \text{rhoi}(n, k, l, g, lp, gp) \wedge (k \in \text{nset}(n)) \wedge \text{at}(lp, k, \texttt{8}))$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11 12)})$

```
;;;If the k's entry in lp is between 9 and 12,
;;;then  the k's entry in l is between 8 and 11.
```

THEOREM: lp9-12-k-in-l8-11
$(\text{ws}(n, l, g)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)}))$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(8 9 10 11)})$

```
;;;If the k's entry in lp is between 9 and 12,
;;;then the k's entry in l is between 5 and 12.
```

THEOREM: k-in-lp9-12-then-l5-12
$(\text{ws}(n, l, g)$
$\wedge \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\wedge \quad (k \in \text{nset}(n))$
$\wedge \quad \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)}))$
$\rightarrow \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11 12)})$

```
;;;If the k's entry in lp is between 8 and 12,
;;;then the k's entry in l is between 5 and 12.
```

Theorem: k-in-l5-12
(ws (n, l, g)
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(k \in$ nset $(n))$
 ∧ union-at-n $(lp, k,$ `'(8 9 10 11 12)`$))$
→ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11 12)`$)$

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds
;;;and k is not its witness, then the witness has
;;;its entry in l between 5 and 12.
;;;ex-lp8-12-in-lp8-12, member-ex-union used.
```

Theorem: k-neq-ex-lp8-12-in-l5-12
(ws (n, l, g)
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(k \in$ nset $(n))$
 ∧ exist-union $(lp, n,$ `'(8 9 10 11 12)`$)$
 ∧ $(k \neq$ exist-union $(lp, n,$ `'(8 9 10 11 12)`$)))$
→ union-at-n $(l,$
  exist-union $(lp, n,$ `'(8 9 10 11 12)`$)$,
  `'(5 6 7 8 9 10 11 12)`$)$

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds and
;;;the witness has its entry in lp between 8 and 12,
;;;then its entry in l is between 5 and 12.
;;;ex-lp8-12-in-lp8-12, member-ex-union used.
```

Theorem: ex-lp8-12-then-l5-12
(ws (n, l, g)
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(k \in$ nset $(n))$
 ∧ exist-union $(lp, n,$ `'(8 9 10 11 12)`$))$
→ union-at-n $(l,$
  exist-union $(lp, n,$ `'(8 9 10 11 12)`$)$,
  `'(5 6 7 8 9 10 11 12)`$)$

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds
;;;and k is not the witness of
;;;(exist-union lp n '(8 9 10 11 12)), then
;;;the witness has its entry 4 in gp.
```

Theorem: ex-lp8-12-in-gp4
(ws (n, l, g)
 ∧ $(k \in$ nset $(n))$
 ∧ rhoi (n, k, l, g, lp, gp)

∧ at (l, k, `12`)
∧ a3-at-n1-n2 (k, exist-union (lp, n, `'(8 9 10 11 12)`), l, g)
∧ ($k \neq$ exist-union (lp, n, `'(8 9 10 11 12)`))
∧ exist-union (lp, n, `'(8 9 10 11 12)`))
→ at (gp, exist-union (lp, n, `'(8 9 10 11 12)`), 4)

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds and
;;;k's entry in l is 12 then the witness has its
;;;entry in lp between 8 and 12 and in gp either 3 or 4.
```

THEOREM: lm-a1-k-in-l12-nep-l8-12
(ws (n, l, g)
∧ ($k \in$ nset (n))
∧ rhoi (n, k, l, g, lp, gp)
∧ exist-union (lp, n, `'(8 9 10 11 12)`)
∧ at (l, k, `12`)
∧ a3-at-n1-n2 (k, exist-union (lp, n, `'(8 9 10 11 12)`), l, g))
→ union-at-n (gp, exist-union (lp, n, `'(8 9 10 11 12)`), `'(3 4)`)

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds
;;;and k's entry in l is 12, then
;;;(exist-intersect-8-12-3-4 n lp gp) holds.
```

THEOREM: a1-k-in-l12-nep-l8-12
(ws (n, l, g)
∧ ($k \in$ nset (n))
∧ rhoi (n, k, l, g, lp, gp)
∧ exist-union (lp, n, `'(8 9 10 11 12)`)
∧ at (l, k, `12`)
∧ a3-at-n1-n2 (k, exist-union (lp, n, `'(8 9 10 11 12)`), l, g))
→ exist-intersect-8-12-3-4 (n, lp, gp)

```
;;;Auxiliary lemma for a1-nep-l8-12.
;;;We have an instance of a3 in the lemma.
```

THEOREM: lm1-a1-nep-l8-12
(ws (n, l, g)
∧ ($k \in$ nset (n))
∧ rhoi (n, k, l, g, lp, gp)
∧ lg (n, l, g)
∧ a1 (n, l, g)
∧ a3-at-n1-n2 (k, exist-union (lp, n, `'(8 9 10 11 12)`), l, g)
∧ exist-union (lp, n, `'(8 9 10 11 12)`)
∧ exist-union (l, n, `'(8 9 10 11 12)`))
→ exist-intersect-8-12-3-4 (n, lp, gp)

THEOREM: lm-a1-nep-l8-12
(ws $(n,\ l,\ g)$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a1 $(n,\ l,\ g)$
 ∧ a3 $(n,\ n,\ l,\ g)$
 ∧ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$)$
 ∧ exist-union $(lp,\ n,$ `'(8 9 10 11 12)`$))$
→ exist-intersect-8-12-3-4 $(n,\ lp,\ gp)$

```
;;;If (exist-union lp n '(8 9 10 11 12)) and
;;;(exist-union l n '(8 9 10 11 12)) hold,
;;;then so does (exist-intersect-8-12-3-4 n lp gp).
```

THEOREM: a1-nep-l8-12
(ws $(n,\ l,\ g)$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a1 $(n,\ l,\ g)$
 ∧ a3 $(n,\ n,\ l,\ g)$
 ∧ exist-union $(l,\ n,$ `'(8 9 10 11 12)`$))$
→ a1 $(n,\ lp,\ gp)$

```
;;;If (exist-union lp n '(8 9 10 11 12)) holds,
;;;then so does (exist-intersect-8-12-3-4 n lp gp).
```

THEOREM: rho-preserves-a1
(ws $(n,\ l,\ g)$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a1 $(n,\ l,\ g)$
 ∧ a3 $(n,\ n,\ l,\ g))$
→ a1 $(n,\ lp,\ gp)$

```
;;;;;;;;;;;;;;;;;;;;;;;;;;;; a2.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;* i-eq-k-j-neq-k

;;;If the k's entry in lp is between 10 and 12
;;;and the k's entry in l is between 10 and 12,
;;; then (phi9 k n g) holds.
```

THEOREM: k-in-l10-11-or-phi9

(ws (n, l, g)
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ union-at-n $(lp, k,$ `'(10 11 12)`$)$
 ∧ $(\neg$ union-at-n $(l, k,$ `'(10 11)`$)))$
→ phi9 (k, n, g)

```
;;;If j is less than k and (phi9 k n g) holds,
;;;then the j's entry in g is either 0 or 1.
```

Theorem: phi9-j-in-g01

$((j \in \text{nset}(n)) \wedge (j < k) \wedge \text{phi9}(k, n, g))$
→ union-at-n $(g, j,$ `'(0 1)`$)$

```
;;;If j is less than k and (phi9 k n g) holds,
;;;then the j's entry in lp is not between 5 and 12.
;;;lp-same-l-not is used.
```

Theorem: case-k-in-phi9

(ws (n, l, g)
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(j \neq k)$
 ∧ $(j < k)$
 ∧ lg (n, l, g)
 ∧ phi9 $(k, n, g))$
→ $(\neg$ union-at-n $(lp, j,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;If j is not equal to k and the k's entry in l is
;;;either 10 or 11, then the j's entry in lp is not
;;;between 5 and 12.
```

Theorem: case-k-in-l10-11

(ws (n, l, g)
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ a2-at-n1-n2 (k, j, l)
 ∧ $(j \neq k)$
 ∧ union-at-n $(l, k,$ `'(10 11)`$))$
→ $(\neg$ union-at-n $(lp, j,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;Auxiliary lemma for lm-i-eq-k-j-neq-k with
;;;(a2-at-n1-n2 k j l).
```

THEOREM: lm1-i-eq-k-j-neq-k
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(j \neq k)$
 ∧ $(j < k)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a2-at-n1-n2 $(k,\ j,\ l)$
 ∧ union-at-n $(lp,\ k,$ `'(10 11 12)`))
→ (¬ union-at-n $(lp,\ j,$ `'(5 6 7 8 9 10 11 12)`))

```
;;;If j is less then k and the k's entry in lp is
;;;between 10 and 12, then the j's entry in lp is
;;;not between 5 and 12.
```

THEOREM: lm-i-eq-k-j-neq-k
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(j \neq k)$
 ∧ $(j < k)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a2-at-n1-n2 $(k,\ j,\ l)$)
→ a2-at-n1-n2 $(k,\ j,\ lp)$

```
;;;If j is less than k,
;;;then (a2-at-n1-n2 k j lp) holds.
```

THEOREM: i-eq-k-j-neq-k
(ws $(n,\ l,\ g)$
 ∧ $(j \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(j \neq k)$
 ∧ $(j < k)$
 ∧ lg $(n,\ l,\ g)$
 ∧ a2 $(n,\ n,\ l)$)
→ a2-at-n1-n2 $(k,\ j,\ lp)$

```
;* j-eq-k-i-neq-k

;;;If the k's entry in l is not 4 and the k's entry in lp
;;;is between 5 and 7, then the k's entry in l is
;;;between 5 and 7.
```

THEOREM: k-in-lp5-7-not-l4-then-l5-7
$(\mathrm{ws}(n, l, g)$
$\land \quad (k \in \mathrm{nset}(n))$
$\land \quad \mathrm{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (\neg\, \mathrm{at}(l, k, 4))$
$\land \quad \text{union-at-n}(lp, k, \texttt{'(5 6 7)}))$
$\to \quad \text{union-at-n}(l, k, \texttt{'(5 6 7)})$

```
;;;If the k's entry in lp is between 5 and 7 then
;;;the k's entry in l is  certainly between 5 and 12.
```

THEOREM: k-in-lp5-7-then-l5-11
$(\mathrm{ws}(n, l, g)$
$\land \quad (k \in \mathrm{nset}(n))$
$\land \quad \mathrm{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (\neg\, \mathrm{at}(l, k, 4))$
$\land \quad \text{union-at-n}(lp, k, \texttt{'(5 6 7)}))$
$\to \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11)})$

```
;;;If the k's entry in lp is 8,
;;;then the k's entry in l is between 5 and 11.
```

THEOREM: k-in-lp8-then-l5-11
$(\mathrm{ws}(n, l, g) \land (k \in \mathrm{nset}(n)) \land \mathrm{rhoi}(n, k, l, g, lp, gp) \land \mathrm{at}(lp, k, 8))$
$\to \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11)})$

```
;;;If the k's entry in lp is between 9 and 12,
;;;then the k's entry in l is between 5 and 12.
```

THEOREM: k-in-lp9-12-then-l5-11
$(\mathrm{ws}(n, l, g)$
$\land \quad (k \in \mathrm{nset}(n))$
$\land \quad \mathrm{rhoi}(n, k, l, g, lp, gp)$
$\land \quad \text{union-at-n}(lp, k, \texttt{'(9 10 11 12)}))$
$\to \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11)})$

```
;;;If the k's entry in l is not 4 an the k's entry in lp is
;;;between 5 and 12, then the k's entry in l is
;;;between 5 and 11.
```

THEOREM: k-in-l5-11
$(\mathrm{ws}(n, l, g)$
$\land \quad (k \in \mathrm{nset}(n))$
$\land \quad \mathrm{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (\neg\, \mathrm{at}(l, k, 4))$
$\land \quad \text{union-at-n}(lp, k, \texttt{'(5 6 7 8 9 10 11 12)}))$
$\to \quad \text{union-at-n}(l, k, \texttt{'(5 6 7 8 9 10 11)})$

```
;;;If the k's entry in l is not 4, and the k's entry
;;;in lp is not between 5 and 12, then the k's entry
;;;in lp is not between 5 and 12.
```

THEOREM: k-not-in-l4
(ws $(n,\ l,\ g)$
 ∧ $(k \in$ nset $(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(\neg$ at $(l,\ k,\ 4))$
 ∧ $(\neg$ union-at-n $(l,\ k,$ `'(5 6 7 8 9 10 11 12)`$)))$
→ $(\neg$ union-at-n $(lp,\ k,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;If a0 holds, and the k's entry in l is not
;;;between 5 and 12, then the k's entry in lp is not
;;;between 5 and 12.
```

THEOREM: k-not-in-lp5-12
(ws $(n,\ l,\ g)$
 ∧ $(i \in$ nset $(n))$
 ∧ $(k \in$ nset $(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ a0 $(n,\ l,\ k)$
 ∧ union-at-n $(l,\ i,$ `'(10 11 12)`$)$
 ∧ $(\neg$ union-at-n $(l,\ k,$ `'(5 6 7 8 9 10 11 12)`$)))$
→ $(\neg$ union-at-n $(lp,\ k,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;Auxiliary lemma for lm-i-neq-k-j-eq-k.
;;;There is (a2-at-n1-n2 i k l) in the lemma.
```

THEOREM: lm1-i-neq-k-j-eq-k
(ws $(n,\ l,\ g)$
 ∧ $(i \in$ nset $(n))$
 ∧ $(k \in$ nset $(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(i \neq k)$
 ∧ $(k < i)$
 ∧ a0 $(n,\ l,\ k)$
 ∧ a2-at-n1-n2 $(i,\ k,\ l)$
 ∧ union-at-n $(lp,\ i,$ `'(10 11 12)`$))$
→ $(\neg$ union-at-n $(lp,\ k,$ `'(5 6 7 8 9 10 11 12)`$))$

```
;;;If k is less than i and the i's entry in lp is
;;;between 10 and 12, then the k's entry in lp is
;;;between 5 and 12.
```

THEOREM: lm-i-neq-k-j-eq-k
(ws $(n,\ l,\ g)$
 ∧ $(i \in \text{nset}\,(n))$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(i \neq k)$
 ∧ $(k < i)$
 ∧ a0 $(n,\ l,\ k)$
 ∧ a2-at-n1-n2 $(i,\ k,\ l))$
 → a2-at-n1-n2 $(i,\ k,\ lp)$

```
;;;If k is less than i then (a2-at-n1-n2 i k lp) holds.
```

THEOREM: i-neq-k-j-eq-k
(ws $(n,\ l,\ g)$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ $(i \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(i \neq k)$
 ∧ $(k < i)$
 ∧ a0 $(n,\ l,\ k)$
 ∧ a2 $(n,\ n,\ l))$
 → a2-at-n1-n2 $(i,\ k,\ lp)$

```
;* i-j-neq-k

;;;If i and j are not equal to k and the i's entry in lp is
;;;between 10 and 12, then the j's entry in lp is
;;;between 5 and 12.
```

THEOREM: lm-i-j-neq-k
(ws $(n,\ l,\ g)$
 ∧ $(i \in \text{nset}\,(n))$
 ∧ $(j \in \text{nset}\,(n))$
 ∧ $(k \in \text{nset}\,(n))$
 ∧ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 ∧ $(i \neq k)$
 ∧ $(j \neq k)$
 ∧ $(j < i)$
 ∧ a2-at-n1-n2 $(i,\ j,\ l))$
 → a2-at-n1-n2 $(i,\ j,\ lp)$

```
;;;If i and j are not equal to k,
;;; then (a2-at-n1-n2 i j lp) holds.
```

Theorem: i-j-neq-k

$$
\begin{aligned}
&(\mathrm{ws}\,(n,\ l,\ g)\\
&\ \wedge\ \ (i \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ (j \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ (k \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp)\\
&\ \wedge\ \ (i \neq k)\\
&\ \wedge\ \ (j \neq k)\\
&\ \wedge\ \ (j < i)\\
&\ \wedge\ \ \mathrm{a2}\,(n,\ n,\ l))\\
&\rightarrow\ \ \text{a2-at-n1-n2}\,(i,\ j,\ lp)
\end{aligned}
$$

```
;;;If i is not equal to k and j is less than i,
;;;then (a2-at-n1-n2 i j lp) holds.
;;;The order of the hints is crucial.
```

Theorem: i-neq-k

$$
\begin{aligned}
&(\mathrm{ws}\,(n,\ l,\ g)\\
&\ \wedge\ \ (k \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ (i \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ (j \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp)\\
&\ \wedge\ \ (i \neq k)\\
&\ \wedge\ \ (j < i)\\
&\ \wedge\ \ \mathrm{a0}\,(n,\ l,\ k)\\
&\ \wedge\ \ \mathrm{a2}\,(n,\ n,\ l))\\
&\rightarrow\ \ \text{a2-at-n1-n2}\,(i,\ j,\ lp)
\end{aligned}
$$

```
;;;If j is less than k then (a2-at-n1-n2 k j lp) holds.
```

Theorem: i-eq-k

$$
\begin{aligned}
&(\mathrm{ws}\,(n,\ l,\ g)\\
&\ \wedge\ \ (j \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ (k \in \mathrm{nset}\,(n))\\
&\ \wedge\ \ \mathrm{rhoi}\,(n,\ k,\ l,\ g,\ lp,\ gp)\\
&\ \wedge\ \ (j < k)\\
&\ \wedge\ \ \mathrm{lg}\,(n,\ l,\ g)\\
&\ \wedge\ \ \mathrm{a2}\,(n,\ n,\ l))\\
&\rightarrow\ \ \text{a2-at-n1-n2}\,(k,\ j,\ lp)
\end{aligned}
$$

```
;;;If i is less than j then (a2-at-n1-n2 k j lp) holds.
;;;Again the order of the hints is crucial.
```

Theorem: rho-preserves-a2

$(\mathrm{ws}\,(n,\ l,\ g)$

$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $(i \in \text{nset}(n))$
$\wedge$ $(j \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $(j < i)$
$\wedge$ $\text{lg}(n, l, g)$
$\wedge$ $\text{a0}(n, l, k)$
$\wedge$ $\text{a2}(n, n, l))$
$\rightarrow$ $\text{a2-at-n1-n2}(i, j, lp)$

```
;;;;;;;;;;;;;;;;;;;;;;;;;;; a3.ev ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;* j-eq-k-i-neq-k

;;;If the i's entry in l is 12 and the k's entry in lp is
;;;between 5 and 12 then the k's entry in l is between 9
;;;and 11.
```

THEOREM: lm-k-in-l9-11
$(\text{ws}(n, l, g)$
$\wedge$ $(i \in \text{nset}(n))$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $\text{lg}(n, l, g)$
$\wedge$ $\text{a0}(n, l, k)$
$\wedge$ $\text{a3-at-n1-n2}(i, k, l, g)$
$\wedge$ at $(l, i,$ `12`$)$
$\wedge$ union-at-n $(l, k,$ `'(5 6 7 8 9 10 11)`$))$
$\rightarrow$ union-at-n $(l, k,$ `'(9 10 11)`$)$

```
;;;If i is not equal to k, the i's entry in l is 12,
;;;and the k's entry in lp is between 5 and 12,
;;;then the k's entry in l is between 9 and 11.
```

THEOREM: k-in-l9-11
$(\text{ws}(n, l, g)$
$\wedge$ $(i \in \text{nset}(n))$
$\wedge$ $(k \in \text{nset}(n))$
$\wedge$ $\text{rhoi}(n, k, l, g, lp, gp)$
$\wedge$ $\text{lg}(n, l, g)$
$\wedge$ $\text{a0}(n, l, k)$
$\wedge$ $\text{a3-at-n1-n2}(i, k, l, g)$
$\wedge$ at $(l, i,$ `12`$)$
$\wedge$ union-at-n $(lp, k,$ `'(5 6 7 8 9 10 11 12)`$))$
$\rightarrow$ union-at-n $(l, k,$ `'(9 10 11)`$)$

```
;;;If the k's entry in lp is between 9 and 11
;;;then the k's entry in lp is between 9 and 12.
```

Theorem: k-in-lp9-12
(ws (n, l, g)
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ union-at-n $(l, k,$ `'(9 10 11)`$))$
 → union-at-n $(lp, k,$ `'(9 10 11 12)`$)$

```
;;Auxiliary lemma for lm-a3-i-neq-k-j-eq-k.
;;;There is (a3-at-n1-n2 i k l g) in the lemma.
```

Theorem: lm1-a3-i-neq-k-j-eq-k
(ws (n, l, g)
 ∧ $(i \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(i \neq k)$
 ∧ lg (n, l, g)
 ∧ lg (n, lp, gp)
 ∧ a0 (n, l, k)
 ∧ a3-at-n1-n2 (i, k, l, g)
 ∧ at $(l, i,$ `12`$)$
 ∧ union-at-n $(lp, k,$ `'(5 6 7 8 9 10 11 12)`$))$
 → at $(gp, k,$ `4`$)$

```
;;;If i is not equal to k, the i's entry in lp is 12,
;;;and the k's entry in lp is between 5 and 12,
;;;then the k's entry in gp is 4.
```

Theorem: lm-a3-i-neq-k-j-eq-k
(ws (n, l, g)
 ∧ $(i \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ $(i \neq k)$
 ∧ lg (n, l, g)
 ∧ a0 (n, l, k)
 ∧ a3-at-n1-n2 $(i, k, l, g))$
 → a3-at-n1-n2 (i, k, lp, gp)

```
;;;If i is not equal to k,
;;;then (a3-at-n1-n2 i k lp gp) holds.
;;;The order of the hypotheses is crucial.
```

THEOREM: a3-i-neq-k-j-eq-k
(ws (n, l, g)
 ∧ $(i \in \text{nset}(n))$
 ∧ $(k \in \text{nset}(n))$
 ∧ rhoi (n, k, l, g, lp, gp)
 ∧ lg (n, l, g)
 ∧ a0 (n, l, k)
 ∧ a3 (n, n, l, g)
 ∧ $(i \neq k))$
→ a3-at-n1-n2 (i, k, lp, gp)

```
;* i-eq-k-j-neq-k

;;;If the k's entry in lp is 12 then (phi11 k n g) holds.
```

THEOREM: cond-rhoi11
(ws (n, l, g) ∧ $(k \in \text{nset}(n))$ ∧ rhoi (n, k, l, g, lp, gp) ∧ at $(lp, k, 12))$
→ phi11 (k, n, g)

```
;;;If the k's entry in l is between 10 and 12,
;;;the j's entry in l is between 5 and 12, and
;;;(a2-at-n2 k n l) holds, then k is less than j.
;;;Because Bmp does not rewrite the clause
;;;(lessp k j), we take its contrapositive.
```

THEOREM: k-lt-j
$((j \in \text{nset}(n))$
 ∧ $(j \neq k)$
 ∧ union-at-n $(l, k,$ '(10 11 12))
 ∧ union-at-n $(l, j,$ '(5 6 7 8 9 10 11 12))
 ∧ $(k \not< j))$
→ (¬ a2-at-n2 $(k, n, l))$

```
;;;If k is less than j and (phi11 k n g) holds,
;;;then the j's entry in g is either 2 or 3.
```

THEOREM: phi11-j-not-in-g23
$((j \in \text{nset}(n))$ ∧ $(k < j)$ ∧ phi11 $(k, n, g))$
→ (¬ union-at-n $(g, j,$ '(2 3)))

```
;;;If j is not equal to k, (a2-at-n2 k n l), (phi11 k n g)
;;;the k's entry in l is between 10 and 12 and
;;;the j's entry in l is between 5 and 12,
;;;then the j's entry in g is either 2 or 3.
```

THEOREM: lm1-j-not-in-g23
$((j \in \text{nset}(n))$
$\land \quad (j \neq k)$
$\land \quad \text{a2-at-n2}(k, n, l)$
$\land \quad \text{phi11}(k, n, g)$
$\land \quad \text{union-at-n}(l, k, \texttt{'(10 11 12)})$
$\land \quad \text{union-at-n}(l, j, \texttt{'(5 6 7 8 9 10 11 12)}))$
$\rightarrow \quad (\neg\ \text{union-at-n}(g, j, \texttt{'(2 3)}))$

```
;;;If j is not equal to k, the k's entry in l is
;;;between 10 and 12, the k's entry in lp is 12 and
;;;the j's entry in l is between 5 and 12,
;;;then the j's entry in g is either 2 or 3.
```

THEOREM: lm2-j-not-in-g23
$(\text{ws}(n, l, g)$
$\land \quad (j \in \text{nset}(n))$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (j \neq k)$
$\land \quad \text{a2-at-n2}(k, n, l)$
$\land \quad \text{at}(lp, k, \texttt{12})$
$\land \quad \text{at}(l, k, \texttt{11})$
$\land \quad \text{union-at-n}(l, j, \texttt{'(5 6 7 8 9 10 11 12)}))$
$\rightarrow \quad (\neg\ \text{union-at-n}(g, j, \texttt{'(2 3)}))$

```
;;;If j is not equal to k, the k's entry in lp is 12,
;;;and the j's entry in l is between 5 and 12,
;;;then the j's entry in g is either 2 or 3.
```

THEOREM: j-not-in-g23
$(\text{ws}(n, l, g)$
$\land \quad (j \in \text{nset}(n))$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (j \neq k)$
$\land \quad \text{a2}(n, n, l)$
$\land \quad \text{at}(l, k, \texttt{11})$
$\land \quad \text{at}(lp, k, \texttt{12})$
$\land \quad \text{union-at-n}(l, j, \texttt{'(5 6 7 8 9 10 11 12)}))$
$\rightarrow \quad (\neg\ \text{union-at-n}(g, j, \texttt{'(2 3)}))$

THEOREM: j-in-g4
$((j \in \text{nset}(n))$
$\land \quad \text{lg}(n, l, g)$

$\wedge$ ($\neg$ union-at-n (g, j, '(2 3)))
$\wedge$ union-at-n (l, j, '(5 6 7 8 9 10 11 12)))
$\rightarrow$ at (g, j, 4)

```
;;;un9-12-then-un8-12, if3, j-not-in-g23,
;;;l5-12-eq-l5-8-or-l9-12, un8-12-then-un5-12,
;;;and j-in-g4 are used.
;;;If j is not equal to k, the k's entry in lp is 12,
;;;the j's entry in l is between 5 and 12,
;;;then the j's entry in g is 4.
```

THEOREM: a3-j-in-l5-12
(ws (n, l, g)
$\wedge$ ($j \in$ nset (n))
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ ($j \neq k$)
$\wedge$ lg (n, l, g)
$\wedge$ a2 (n, n, l)
$\wedge$ at (l, k, 11)
$\wedge$ at (lp, k, 12)
$\wedge$ union-at-n (l, j, '(5 6 7 8 9 10 11 12)))
$\rightarrow$ at (g, j, 4)

```
;;;If the k's entry in lp is 12,
;;;then the k's entry in l is 11.
```

THEOREM: k-in-l11
(ws (n, l, g) $\wedge$ ($k \in$ nset (n)) $\wedge$ rhoi (n, k, l, g, lp, gp) $\wedge$ at (lp, k, 12))
$\rightarrow$ at (l, k, 11)

```
;;;If k is not equal to j and the j's entry in g is 4,
;;;then the j's entry in gp is 4.
```

THEOREM: lm1-a3-i-eq-k-j-neq-k
(ws (n, l, g)
$\wedge$ ($j \in$ nset (n))
$\wedge$ ($k \in$ nset (n))
$\wedge$ rhoi (n, k, l, g, lp, gp)
$\wedge$ ($j \neq k$)
$\wedge$ lg (n, l, g)
$\wedge$ a2 (n, n, l)
$\wedge$ at (lp, k, 12)
$\wedge$ union-at-n (lp, j, '(5 6 7 8 9 10 11 12)))
$\rightarrow$ at (g, j, 4)

```
;;;If j is not equal to k, the k's entry in lp is 12,
;;;and the j's entry in lp is between 5 and 12,
;;;then the j's entry in gp is 4.
```

THEOREM: lm-a3-i-eq-k-j-neq-k
$(\text{ws}(n, l, g)$
$\land \quad (j \in \text{nset}(n))$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (j \neq k)$
$\land \quad \text{lg}(n, l, g)$
$\land \quad \text{a2}(n, n, l)$
$\land \quad \text{a3-at-n1-n2}(k, j, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(k, j, lp, gp)$

```
;;;If j is not equal to k then (a3-at-n1-n2 k j lp gp).
```

THEOREM: a3-i-eq-k-j-neq-k
$(\text{ws}(n, l, g)$
$\land \quad (j \in \text{nset}(n))$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (j \neq k)$
$\land \quad \text{lg}(n, l, g)$
$\land \quad \text{a2}(n, n, l)$
$\land \quad \text{a3}(n, n, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(k, j, lp, gp)$

```
;* i-j-neq-k

;;;If i,j are not equal to k, the i's entry in lp is 12
;;;and the j's entry in lp between 5 and 12
;;;then the j's entry in gp is 4.
```

THEOREM: lm-a3-i-j-neq-k
$(\text{ws}(n, l, g)$
$\land \quad (i \in \text{nset}(n))$
$\land \quad (j \in \text{nset}(n))$
$\land \quad (k \in \text{nset}(n))$
$\land \quad \text{rhoi}(n, k, l, g, lp, gp)$
$\land \quad (i \neq k)$
$\land \quad (j \neq k)$
$\land \quad \text{a3-at-n1-n2}(i, j, l, g))$
$\rightarrow \quad \text{a3-at-n1-n2}(i, j, lp, gp)$

```
;;;If i,j are not equal to k,
;;;then (a3-at-n1-n2 i j lp gp).
```

THEOREM: a3-i-j-neq-k
(ws $(n,\ l,\ g)$
 $\land$ $(i \in \text{nset}\,(n))$
 $\land$ $(j \in \text{nset}\,(n))$
 $\land$ $(k \in \text{nset}\,(n))$
 $\land$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\land$ a3 $(n,\ n,\ l,\ g)$
 $\land$ $(i \neq k)$
 $\land$ $(j \neq k))$
$\rightarrow$ a3-at-n1-n2 $(i,\ j,\ lp,\ gp)$

```
;* i-j-eq-k
```

THEOREM: lm-a3-i-j-eq-k
(ws $(n,\ l,\ g)$
 $\land$ $(k \in \text{nset}\,(n))$
 $\land$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\land$ lg $(n,\ l,\ g)$
 $\land$ a2 $(n,\ n,\ l)$
 $\land$ a3-at-n1-n2 $(k,\ k,\ l,\ g))$
$\rightarrow$ a3-at-n1-n2 $(k,\ k,\ lp,\ gp)$

```
;;;(a3-at-n1-n2 k k lp gp) holds by lg.
```

THEOREM: a3-i-j-eq-k
(ws $(n,\ l,\ g)$
 $\land$ $(k \in \text{nset}\,(n))$
 $\land$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\land$ lg $(n,\ l,\ g)$
 $\land$ a2 $(n,\ n,\ l)$
 $\land$ a3 $(n,\ n,\ l,\ g))$
$\rightarrow$ a3-at-n1-n2 $(k,\ k,\ lp,\ gp)$

```
;;;(a3-at-n1-n2 k j lp gp) holds.
```

THEOREM: a3-i-eq-k
(ws $(n,\ l,\ g)$
 $\land$ $(j \in \text{nset}\,(n))$
 $\land$ $(k \in \text{nset}\,(n))$
 $\land$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
 $\land$ lg $(n,\ l,\ g)$

$\wedge$ a2 $(n,\ n,\ l)$
$\wedge$ a3 $(n,\ n,\ l,\ g))$
$\rightarrow$ a3-at-n1-n2 $(k,\ j,\ lp,\ gp)$

```
;;;If i is not equal to k,
;;;then (a3-at-n1-n2 i j lp gp) holds.
```

THEOREM: a3-i-neq-k
(ws $(n,\ l,\ g)$
$\wedge$ $(k \in$ nset $(n))$
$\wedge$ $(i \in$ nset $(n))$
$\wedge$ $(j \in$ nset $(n))$
$\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
$\wedge$ lg $(n,\ l,\ g)$
$\wedge$ a0 $(n,\ l,\ k)$
$\wedge$ a2 $(n,\ n,\ l)$
$\wedge$ a3 $(n,\ n,\ l,\ g)$
$\wedge$ $(i \neq k))$
$\rightarrow$ a3-at-n1-n2 $(i,\ j,\ lp,\ gp)$

```
;;;(a3-at-n1-n2 i j lp gp) holds.
```

THEOREM: rho-preserves-a3
(ws $(n,\ l,\ g)$
$\wedge$ $(k \in$ nset $(n))$
$\wedge$ $(i \in$ nset $(n))$
$\wedge$ $(j \in$ nset $(n))$
$\wedge$ rhoi $(n,\ k,\ l,\ g,\ lp,\ gp)$
$\wedge$ lg $(n,\ l,\ g)$
$\wedge$ a0 $(n,\ l,\ k)$
$\wedge$ a2 $(n,\ n,\ l)$
$\wedge$ a3 $(n,\ n,\ l,\ g))$
$\rightarrow$ a3-at-n1-n2 $(i,\ j,\ lp,\ gp)$

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