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; 1/26/87

; Here is a mechanically-checked proof of the finite version
; of Ramsey's Theorem for exponent 2.  This proof was shown
; to me (with ordinary mathematical notation rather than in
; the Boyer-Moore logic) by Jim Schmerl, who claims that it's
; not a new proof.

; Theorem (RAMSEY-THM-2):

; (IMPLIES (LEQ (RAMSEY P Q)
;               (LENGTH DOMAIN))
;          (AND
;            (SUBSETP (HOM-SET PAIRS DOMAIN P Q)
;                     DOMAIN)
;            (OR
;              (AND (HOMOGENEOUS (HOM-SET PAIRS DOMAIN P Q)
;                                PAIRS
;                                1)
;                   (LEQ P (LENGTH (HOM-SET PAIRS DOMAIN P Q))))
;              (AND (HOMOGENEOUS (HOM-SET PAIRS DOMAIN P Q)
;                                PAIRS
;                                2)
;                   (LEQ Q (LENGTH (HOM-SET PAIRS DOMAIN P Q)))))))

; Here, HOM-SET can be thought of as a Skolem function which picks
; out the desired homogeneous subset of the given DOMAIN, i.e.
; homogeneous with respect to the relation PAIRS (presented as a
; list of ordered pairs).  Notice that if DOMAIN has no repetitions
; (which is a harmless assumption for the statement of Ramsey's
; Theorem), then LENGTH is just cardinality.  The function LEQ is
; just "less than or equal", at least on the natural numbers.
; (HOMOGENEOUS S PAIRS 1) is true, i.e. equals T, if the list
; S is homogeous with respect to PAIRS in the positive sense,
; i.e., in that each pair from S is related by PAIRS (i.e. either
; the pair or its reverse belongs to PAIRS).  Similary,
; (HOMOGENEOUS S PAIRS 2) is true if no pair from S belongs to
; PAIRS (in the same sense).  RAMSEY is a function which returns
; an integer which (according to the theorem above) is "large
; enough".  However, at the end of this note we prove that
; (RAMSEY P Q) = (CHOOSE (PLUS P Q) P), where (CHOOSE M N) is
; the binomial coefficient (defined though with integer division).
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; The description above completes the discussion of the theorem,
; but to read on it helps to realize that we don't define HOM-SET
; until near the end of this note.  Instead, we define a function
; WIT which returns a pair, so that HOM-SET merely returns the CAR
; of this pair.  The CDR of the value of WIT is the "color", which
; is 1 or 2 (as is proved in a lemma below).  It's really best to
; think of WIT as a Skolem function, though in fact it is defined
; to make the theorem true.
```

EVENT: Start with the initial **nqthm** theory.

DEFINITION:
ramsey $(p,\ q)$
= **if** $p \simeq 0$ **then** 1
  elseif $q \simeq 0$ **then** 1
  else ramsey $(p-1,\ q)$ + ramsey $(p,\ q-1)$ **endif**

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;this is a hint to get the
;definition accepted
```

DEFINITION:
related $(i,\ j,\ pairs) = ((\text{cons}\,(i,\ j) \in pairs) \vee (\text{cons}\,(j,\ i) \in pairs))$

DEFINITION:
partition (*n*, *rest*, *pairs*)
= **if** listp (*rest*)
  then if related (*n*, car (*rest*), *pairs*)
    then cons (cons (car (*rest*), car (partition (*n*, cdr (*rest*), *pairs*))),
      cdr (partition (*n*, cdr (*rest*), *pairs*)))
    else cons (car (partition (*n*, cdr (*rest*), *pairs*)),
      cons (car (*rest*), cdr (partition (*n*, cdr (*rest*), *pairs*)))) **endif**
  else cons (**nil**, **nil**) **endif**

DEFINITION:
length (*lst*)
= **if** listp (*lst*) **then** 1 + length (cdr (*lst*))
  else 0 **endif**

DEFINITION:
wit (*pairs*, *domain*, *p*, *q*)
= **if** $p \simeq 0$ **then** cons (**nil**, 1)
  elseif $q \simeq 0$ **then** cons (**nil**, 2)
  elseif length (car (partition (car (*domain*), cdr (*domain*), *pairs*)))

```
          <  ramsey (p - 1, q)
   then if cdr (wit (pairs,
                     cdr (partition (car (domain), cdr (domain), pairs)),
                     p,
                     q - 1))
           =  1
        then wit (pairs,
                  cdr (partition (car (domain), cdr (domain), pairs)),
                  p,
                  q - 1)
        else cons (cons (car (domain),
                         car (wit (pairs,
                                   cdr (partition (car (domain),
                                                   cdr (domain),
                                                   pairs)),
                                   p,
                                   q - 1))),
                   2) endif
   elseif cdr (wit (pairs,
                    car (partition (car (domain), cdr (domain), pairs)),
                    p - 1,
                    q))
          =  2
   then wit (pairs, car (partition (car (domain), cdr (domain), pairs)), p - 1, q)
   else cons (cons (car (domain),
                    car (wit (pairs,
                              car (partition (car (domain),
                                              cdr (domain),
                                              pairs)),
                              p - 1,
                              q))),
              1) endif
```

Definition:

```
homogeneous1 (n, domain, pairs, flg)
=  if listp (domain)
   then if flg = 1 then related (n, car (domain), pairs)
        else ¬ related (n, car (domain), pairs) endif
        ∧  homogeneous1 (n, cdr (domain), pairs, flg)
   else t endif
```

Definition:

```
homogeneous (domain, pairs, flg)
=  if listp (domain)
```

then homogeneous1 (car ($domain$), cdr ($domain$), $pairs$, flg)
$\wedge$ homogeneous (cdr ($domain$), $pairs$, flg)
else t endif

THEOREM: length-of-partitions
length (cons (n, $domain$))
$=$ (1 + (length (car (partition (n, $domain$, $pairs$)))
$+$ length (cdr (partition (n, $domain$, $pairs$)))))

THEOREM: lessp-length-ramsey-2
(listp ($domain$)
$\wedge$ ($i \not\simeq$ 0)
$\wedge$ ($j \not\simeq$ 0)
$\wedge$ (($i + j$) $\leq$ length ($domain$))
$\wedge$ (length (car (partition (car ($domain$), cdr ($domain$), $pairs$))) $<$ i))
$\rightarrow$ ((length (cdr (partition (car ($domain$), cdr ($domain$), $pairs$))) $<$ j) $=$ **f**)

THEOREM: ramsey-not-zerop
ramsey (p, q) $\not\simeq$ 0

EVENT: Disable lessp-length-ramsey-2.

THEOREM: lessp-length-ramsey
(($p \not\simeq$ 0)
$\wedge$ ($q \not\simeq$ 0)
$\wedge$ (ramsey (p, q) $\leq$ length ($domain$))
$\wedge$ (length (car (partition (car ($domain$), cdr ($domain$), $pairs$)))
$<$ ramsey ($p - 1$, q)))
$\rightarrow$ ((length (cdr (partition (car ($domain$), cdr ($domain$), $pairs$)))
$<$ ramsey (p, $q - 1$))
$=$ **f**)

THEOREM: wit-flag
(cdr (wit ($pairs$, $domain$, p, q)) $\neq$ 1) $\rightarrow$ (cdr (wit ($pairs$, $domain$, p, q)) $=$ 2)

THEOREM: homogeneous1-partition-cdr
homogeneous1 (a, cdr (partition (a, $domain$, $pairs$)), $pairs$, 2)

THEOREM: homogeneous1-partition-car
homogeneous1 (a, car (partition (a, $domain$, $pairs$)), $pairs$, 1)

DEFINITION:
subsetp (x, y)
$=$ **if** $x \simeq$ **nil then t**
elseif car (x) $\in y$ **then** subsetp (cdr (x), y)
else f endif

THEOREM: subsetp-preserves-homogeneous1
(subsetp $(l,\ m)$ $\wedge$ homogeneous1 $(a,\ m,\ pairs,\ flg)$)
$\rightarrow$ homogeneous1 $(a,\ l,\ pairs,\ flg)$

```
; The following lemmas, till further notice, are simply
; lemmas about sets, some of which may be needed below.
```

THEOREM: member-cons
$(a \in l) \rightarrow (a \in \text{cons}\,(x,\ l))$

THEOREM: subsetp-cons
subsetp $(l,\ m) \rightarrow$ subsetp $(l,\ \text{cons}\,(a,\ m))$

THEOREM: subsetp-reflexivity
subsetp $(x,\ x)$

THEOREM: cdr-subsetp
subsetp (cdr $(x),\ x$)

THEOREM: member-subsetp
$((x \in y) \wedge \text{subsetp}\,(y,\ z)) \rightarrow (x \in z)$

THEOREM: subsetp-is-transitive
(subsetp $(x,\ y)$ $\wedge$ subsetp $(y,\ z)$) $\rightarrow$ subsetp $(x,\ z)$

```
; end of silly subset lemmas
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THEOREM: partition-subsetp
subsetp (car (partition $(a,\ dom,\ pairs)$), dom)
$\wedge$ subsetp (cdr (partition $(a,\ dom,\ pairs)$), dom)

THEOREM: ramsey-not-zero
ramsey $(p,\ q) \neq$ 0

THEOREM: witness-subsetp
(ramsey $(p,\ q) \leq$ length $(domain)$)
$\rightarrow$ subsetp (car (wit $(pairs,\ domain,\ p,\ q)$), $domain$)

```
; Here are the commands to generate proofs of four of
; the six induction steps in the theorem.
```

THEOREM: ramsey-thm-2-goals-1-2-3-5
t

THEOREM: all-but-goal-4
t

THEOREM: ramsey-thm-2-except-length
(ramsey $(p,\ q) \leq$ length $(domain)$)
$\rightarrow$ homogeneous (car (wit $(pairs,\ domain,\ p,\ q)$),
$pairs$,
cdr (wit $(pairs,\ domain,\ p,\ q)$))

THEOREM: ramsey-thm-2-length
(ramsey $(p,\ q) \leq$ length $(domain)$)
$\rightarrow$ ((length (car (wit $(pairs,\ domain,\ p,\ q)$))
$<$ **if** cdr (wit $(pairs,\ domain,\ p,\ q)$) = `1` **then** p
else q **endif**)
$=$ **f**)

```
; Finally, the main theorem's proof is concluded:
```

DEFINITION:
hom-set $(pairs,\ domain,\ p,\ q)$ = car (wit $(pairs,\ domain,\ p,\ q)$)

THEOREM: ramsey-thm-2
(ramsey $(p,\ q) \leq$ length $(domain)$)
$\rightarrow$ (subsetp (hom-set $(pairs,\ domain,\ p,\ q)$, $domain$)
$\wedge$ ((homogeneous (hom-set $(pairs,\ domain,\ p,\ q)$, $pairs$, `1`)
$\wedge$ ($p \leq$ length (hom-set $(pairs,\ domain,\ p,\ q)$)))
$\vee$ (homogeneous (hom-set $(pairs,\ domain,\ p,\ q)$, $pairs$, `2`)
$\wedge$ ($q \leq$ length (hom-set $(pairs,\ domain,\ p,\ q)$)))))

```
;;; The following were aded 4/22/88 to show that in fact, under the
;;; hypotheses of the theorem above, the HOM-SET has no duplicates
;;; (and hence LENGTH really does measure cardinality, though I don't
;;; formalize or prove this parenthetical remark).
```

DEFINITION:
setp (x)
= **if** listp (x) **then** (car $(x) \notin$ cdr (x)) $\wedge$ setp (cdr (x))
else t endif

THEOREM: set-partition
setp (x)
$\rightarrow$ (setp (car (partition $(a,\ x,\ pairs)$)) $\wedge$ setp (cdr (partition $(a,\ x,\ pairs)$)))

THEOREM: member-partition

$$(a \notin x) \rightarrow \quad ((a \notin \text{car}(\text{partition}(b,\ x,\ \textit{pairs}))) \land \quad (a \notin \text{cdr}(\text{partition}(b,\ x,\ \textit{pairs}))))$$

THEOREM: member-cons-expand

$$(a \in \text{cons}(b,\ y)) = \quad \textbf{if } a = b \textbf{ then t} \ \textbf{else } a \in y \textbf{ endif}$$

THEOREM: setp-cons-expand

$$\text{setp}(\text{cons}(b,\ y)) = \quad \textbf{if } b \in y \textbf{ then f} \ \textbf{else } \text{setp}(y) \textbf{ endif}$$

THEOREM: member-hom-set-implies-member-domain

$$((\text{length}(\textit{domain}) \not< \text{ramsey}(p,\ q)) \land (a \notin \textit{domain})) \rightarrow \quad (a \notin \text{car}(\text{wit}(\textit{pairs},\ \textit{domain},\ p,\ q)))$$

THEOREM: setp-hom-set

$$((\text{length}(\textit{domain}) \not< \text{ramsey}(p,\ q)) \land \text{setp}(\textit{domain})) \rightarrow \quad \text{setp}(\text{hom-set}(\textit{pairs},\ \textit{domain},\ p,\ q))$$

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