On a theorem by Lambek and Moser.

While looking for interesting functions mapping sequences of integers into sequences of integers, W.H.J. Feijen found in [1] the following results attributed to J. Lambek and L. Moser.

Let f be an ascending continued concatenation of natural numbers, more precisely:

$$(\underline{A}\, x : x \geq 0 :\ 0 \leq f\,x \leq f\,(x+1)) \quad ,$$

such that, furthermore, $f\,x$ exceeds any given bound F for sufficiently large x, more precisely:

$$(\underline{A}\, F : F \geq 0 : (\underline{E}\, X : X \geq 0 : (\underline{A}\, x : x \geq X : f\,x > F))) \ .$$

Let the function C with $g = Cf$ be defined by the relation for all $y \geq 0$

$$g\,y = (\underline{N}\, x : x \geq 0 :\ f\,x \leq y)$$

(the right-hand side being read as "the number of distinct natural values x such that $f\,x$ is at most y"). Obviously, g is also an ascending continued concatenation such that $g\,y$ exceeds any given bound G for sufficiently large y.

The first result attributed to Lambek and Moser is that then $f = Cg$, as is illustrated by the example

	0	1	2	3	4	5	6	7	8	9	......
f:	0	1	1	3	6	6	7	8	10	10	....
g:	1	3	3	4	4	4	6	7	8	8	....

Let for any continued concatenation q of natural numbers the function D be defined by the relation for all $n \geq 0$

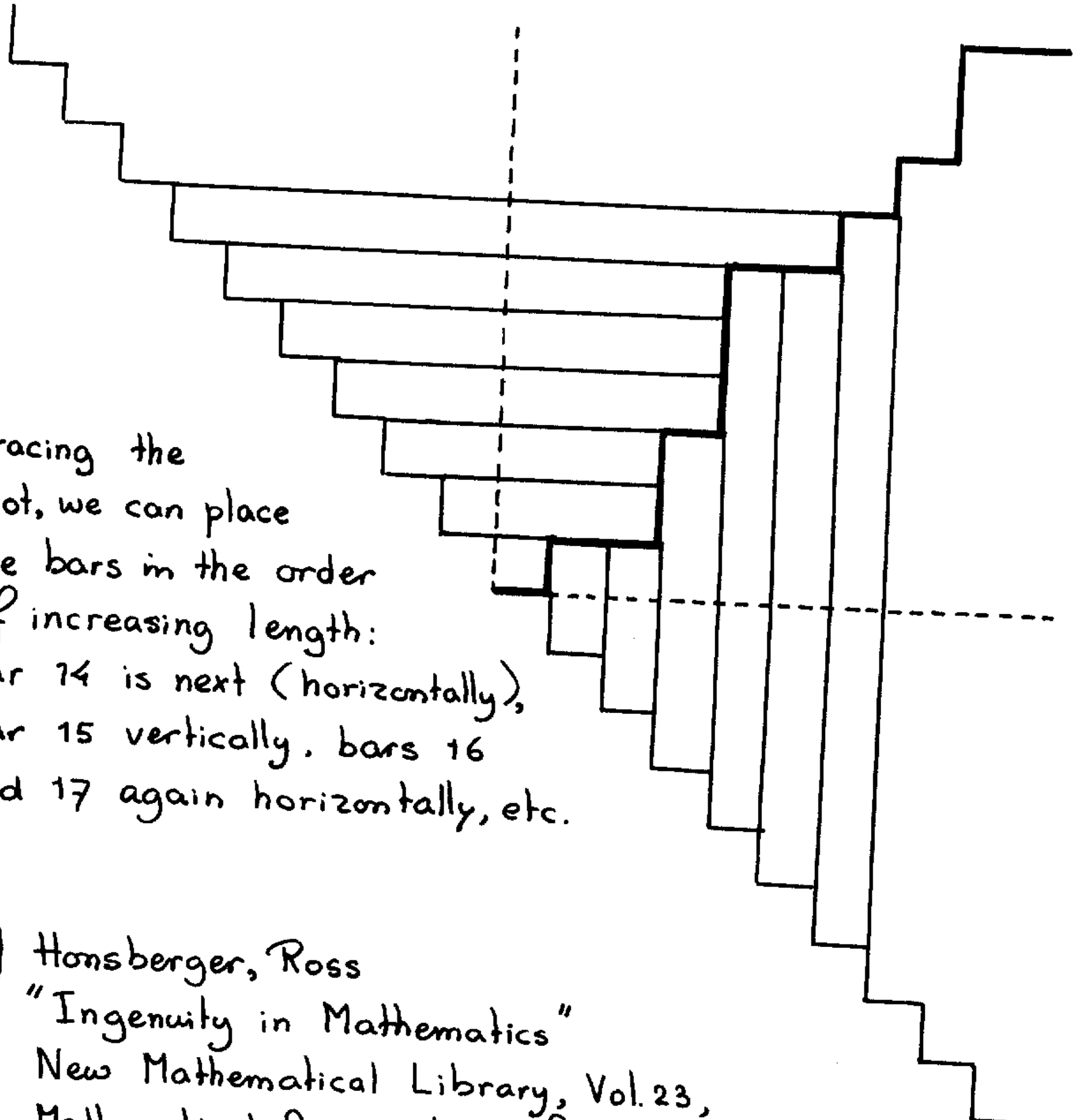
$$D\,q\,n = n + (q\,n) \quad .$$

In the above example we would have

	0	1	2	3	4	5	6	7	8	9	......
Df:	0	2	3	6	10	11	13	15	18	19	....
Dg:	1	4	5	7	8	9	12	14	16	17	....

The second result attributed to Lambek and Moser is that Df and Dg form a partitioning of the natural numbers.

For people willing to generalize pictures the visualization "proves" these results. The elements of f and Df have been represented by vertical bars, one wide and of the corresponding length; those of g and Dg have been represented by horizontal ones. The mapping C then becomes a reflection.



Tracing the plot, we can place the bars in the order of increasing length: bar 14 is next (horizontally), bar 15 vertically, bars 16 and 17 again horizontally, etc.

[1] Honsberger, Ross
"Ingenuity in Mathematics"
New Mathematical Library, Vol. 23,
Mathematical Association of America (1970)

Plataanstraat 5
5671 AL NUENEN
The Netherlands

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prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow