A Hungarian problem.

Martin Rem returned from his trip to Hungary with the following problem. Let two positive integers p and q be given. We start with a bag containing a finite number of integers. When the bag contains some number, x say, at least twice, a move is possible; the move consists of replacing in the bag two occurrences of x by x+p and x-q respectively. Show that this game terminates.

When we characterize the contents of the bag by its characteristic function f — i.e. for all x, f(x)= the number of occurrences of the value x in the bag — and use the initializing guard — which is true when the equation can be solved and sets the unknown to a solution — the game can be written as follows

$$\underline{\text{do}}\ x : f(x) \geq 2 \rightarrow$$
$$f(x-q), f(x), f(x+p) := f(x-q)+1, f(x)-2, f(x+p)+1$$
$$\underline{\text{od}}$$

* * *

To start with, we generalize the game: for each move a new pair (p,q) of positive numbers may be chosen. Furthermore we extend our program with auxiliary array n, initially satisfying for all x f(x) = n(x), and modified by extending the guarded command with the statement

$$n(x-q), n(x+p) := n(x-q)+1, n(x+p)+1 \quad .$$

In other words, after initialization $n(x)$ counts how many times $f(x)$ is increased.

With c defined as $c = (\underline{N}\, x :: n(x) > 0)$ we have

<u>Lemma 0.</u>
Each $n(x)$ is bounded <u>or</u> c is unbounded .

<u>Proof</u>. Since each $f(x)$ is bounded by the constant number of integers in the bag, an $f(x)$ that is increased an unbounded number of times is also decreased an unbounded number of times, causing increases of $n(y)$ for some $y>x$ (and some $y<x$). Hence we have for all x:
$n(x)$ is bounded <u>or</u> $(\underline{S}\, y : y>x : n(y))$ is unbounded .
Hence, for bounded c there is no largest x for which $n(x)$ is unbounded, hence there is no such x at all. (End of Proof of Lemma 0.)

Since "c is bounded <u>and</u> each $n(x)$ is bounded" is equivalent with "the game terminates", and since $P \vee Q$ equals $(\neg Q \wedge P) \vee Q$, we derive from Lemma 0:

<u>Lemma 1.</u>
The game terminates <u>or</u> c is unbounded.

The smallest value in the bag is the smallest value

x such that $n(x) > 0$; the largest value in the bag is the largest value x such that $n(x) > 0$. Hence, from Lemma 1 we conclude

Lemma 2

The game terminates *or* the difference between largest and smallest value in the bag is unbounded.

With the derivation of Lemma 2, the auxiliary array n has done its work. Defining "a gap of length h" as a solution of the equation

$$f(x) > 0 \ \underline{\text{and}}\ f(x+h) > 0 \ \underline{\text{and}}\ (\underline{A}\, j: x < j < x+h: f(j) = 0)$$

we conclude from Lemma 2, because the number of integers in the bag is constant,

Lemma 3

The game terminates *or* the length of the largest gap is unbounded.

Finally we undo some of our generalization: let the choice of the pair of positive numbers for each move be restricted to pairs (p, q) such that $p + q \leq K$ for some K. We then have

Lemma 4

No gap exceeds the maximum of K and the length of the largest initial gap.

Combination of the last two lemmata solves the Hungarian problem.

<u>Acknowledgement and concluding remarks.</u>

Thanks are due to the members of the Tuesday Afternoon Club with whom I discussed my first proof for Lemma 2. That proof used mathematical induction on the number of integers in the bag. I am particularly grateful to R.W. Bulterman for the persistence of his dislike of the three-case analysis the induction step required; he also objected to some handwaving with which I draw a correct but sweeping conclusion from "A possible move remains possible until done".

I must admit that I would have liked a simpler termination proof than the one I found this evening. Perhaps it doesn't exist: Hungary is a traditional source of non-trivial problems.

Considering first the generalized problem was an improvement; relaxing constraints could be a general technique for doing away with reductiones ad absurdum.

Plataanstraat 5
5671 AL NUENEN
The Netherlands

2 December 1980
prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow