Concerning the equivalence

The boolean operators $\wedge$, $\vee$, and $\neg$ are easily expressed in Western languages, e.g. "and", "or" and "not" respectively. Also the implication $\Rightarrow$ can be accommodated, usually by "if ... then". The equivalence $\equiv$ is rendered by "if and only if", but this an artefact of mathematical prose: people are definitely uncomfortable with it, as is shown by the fact that in pronouncing $A \equiv B$, verbal constructs like "holds" or "is true" and quotes are often inserted: "A" holds if and only if "B" holds. This linguistical clumsiness was for one of my mathematical colleagues proof that the equivalence did not come naturally and reason to warn against its use in reasoning.

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We consider variables ranging from 0 to 1; if you so desire you may interpret them "fractions of falsity" or "degree of unlikeliness". We identify 0 with true and 1 with false.

Let $X \wedge Y$ denote X max Y, and let $X \vee Y$ denote X min Y. All sorts of familiar relations hold true, such as

$$X \wedge true = X \qquad X \wedge false = false$$
$$X \vee false = X \qquad X \vee true = true$$
$$X \wedge X = X \qquad X \vee X = X$$

$$X \wedge Y = Y \wedge X \qquad X \vee Y = Y \vee X$$
$$X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$$
$$X \vee (Y \vee Z) = (X \vee Y) \vee Z$$
$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$$
$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z) \quad .$$

Now the negation. Let $\neg X$ denote $1 - X$. Again all sorts of familiar formulae may be used:

$$(\neg\neg X) = X$$
$$\neg(X \wedge Y) = \neg X \vee \neg Y \qquad \neg(X \vee Y) = \neg X \wedge \neg Y \quad .$$

We have lost, however, $\neg X \vee X = true$; as this relation is known as "Tertium non datur", this loss should not surprise us: it is the obvious price to pay for continuous logic.

Let $X \equiv Y$ denote the absolute value of the difference between X and Y. Again all sorts of familiar formulae hold true:

$$\neg X \equiv Y \;=\; X \equiv \neg Y \qquad X \equiv Y \;=\; Y \equiv X$$
$$X \equiv Y \;=\; X \wedge Y \equiv X \vee Y$$
$$X \wedge Y \equiv X \;=\; X \vee Y \equiv Y$$
$$X \equiv Y \;=\; (X \wedge Y \equiv X) \wedge (X \wedge Y \equiv Y)$$

In the last formula but 1 we recognize the two expressions for the implication $X \Rightarrow Y$, in the last one the equivalence as "if and only if".

If we take $X \Rightarrow Y$ as short for $X \wedge Y \equiv X$ or for $X \vee Y \equiv Y$ we have

$$X \Rightarrow Y \ = \ \neg Y \Rightarrow \neg X$$
$$(X \Rightarrow Y) \wedge (X \Rightarrow Z) \ = \ X \Rightarrow Y \wedge Z$$
$$(X \Rightarrow Y) \vee (X \Rightarrow Z) \ = \ X \Rightarrow Y \vee Z$$
$$(X \Rightarrow Y) \wedge (Z \Rightarrow Y) \ = \ X \vee Z \Rightarrow Y$$
$$(X \Rightarrow Y) \vee (Z \Rightarrow Y) \ = \ X \wedge Z \Rightarrow Y$$

a wealth of formulae which might explain the popularity of the implication in "natural logic".

For the equivalence, however, we lose a number of formulae:

$(X \equiv Y) \equiv Z$ and $X \equiv (Y \equiv Z)$ in general differ: equivalence has lost its associativity;

$(X \equiv Y) \vee Z$ and $X \vee Z \equiv Y \vee Z$ in general differ: disjunction no longer distributes over equivalence;

$\neg(X \equiv Y)$ and $\neg X \equiv Y$ in general differ: negation and equivalence no longer "commute".

$(X \equiv Y) \wedge (Y \equiv Z) \Rightarrow (X \equiv Y) \wedge (Y \equiv Z) \wedge (Z \equiv X)$
in general differs from true: equivalence is no longer transitive.

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With the exception of the loss of the "Tertium non datur" — which seems to be suspect anyhow — all formulae that lose their validity in the transition from binary to continuous logic involve the equivalence! And that is — see our introduction — precisely the logical connective our natural languages hardly cater for.

The moral is two-fold. Firstly we conclude that, in all probability, "natural logic" is not as binary at all as George Boole would like us to believe. Secondly we conclude that, compared to Boolean algebra, the calculus reflecting our "natural logic" is an unmanageable mess: the calculus is so defective that basing anything on it seems a dead alley. It is certainly not a calculus we can carry out intuitively.

In fact, quite a number of formulae still valid surprised me and I must confess that at this stage I could only verify them by a painful case analysis. (Another way of stating this is by pointing out that this note contains a whole bunch of surprising little theorems of possible interest.)

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