Well-foundedness and lexical coupling.

In AvG88/EWD1079 —Well-foundedness and the transitive closure— we use, as our definition, that well-foundedness of relation R means that for any predicate P (on the domain of R)

(0) $(\underline{A}y :: P.y) \Leftarrow (\underline{A}y :: P.y \Leftarrow (\underline{A}x : xRy : P.x))$.

When reading AvG88/EWD1079, ATAC raised the question —just like we ourselves had done— whether this definition is more convenient than alternative ones, e.g. that all decreasing chains are of finite length. So as an experiment we then set out to prove some more theorems that so far we had only proved using the decreasing-chains definition. This note records the result of the experiment.

<u>Theorem0</u> If the components of a lexical coupling are well-founded, so is the lexical coupling.

<u>Theorem1</u> If a lexical coupling with non-empty domain is well-founded, so are its components.

Denoting both the lexical coupling and its component relations by "<", we define the former for all (x,a) and (y,b) by

(1) $(x,a) < (y,b) \equiv x<y \vee (x=y \wedge a<b)$.

<u>Proof of Theorem 0</u> By (0), our demonstrandum is for any P

(2) $(\underline{A}\, y,b :: P.y.b) \Leftarrow$

$\quad (\underline{A}\, y,b :: P.y.b \Leftarrow (\underline{A}\, x,a : (x,a) < (y,b) : P.x.a))$,

given (0) for the component relations, i.e. for any Q and R,

(3) $(\underline{A}\, y :: Q.y) \Leftarrow (\underline{A}\, y :: Q.y \Leftarrow (\underline{A}\, x : x<y : Q.x))$

(4) $(\underline{A}\, b :: R.b) \Leftarrow (\underline{A}\, b :: R.b \Leftarrow (\underline{A}\, a : a<b : R.a))$.

Starting with the antecedent of (2) we calculate

$(\underline{A}\, y,b :: P.y.b \Leftarrow (\underline{A}\, x,a : (x,a) < (y,b) : P.x.a))$

= { (1) }

$(\underline{A}\, y,b :: P.y.b \Leftarrow (\underline{A}\, x,a : x<y \vee (x=y \wedge a<b) : P.x.a))$

= { range splitting ; one-point rule }

$(\underline{A}\, y,b :: P.y.b \Leftarrow$

$\qquad (\underline{A}\, x,a : x<y : P.x.a) \wedge (\underline{A}\, a : a<b : P.y.a)$

$)$

= { nesting and pred. calc.: preparing distribution }

$(\underline{A}\, y :: (\underline{A}\, b :: (P.y.b \Leftarrow (\underline{A}\, a : a<b : P.y.a)) \Leftarrow$

$\qquad) \qquad (\underline{A}\, x,a : x<y : P.x.a)$

$)$

$\Rightarrow$ { distributing out antecedent under $(\underline{A}\, b::)$; then (4) with $R := P.y$ on remaining quantification over b }

$(\underline{A}\, y :: (\underline{A}\, b :: P.y.b) \Leftarrow (\underline{A}\, x,a : x<y : P.x.a) \;)$

=

$= \quad \{\text{nesting, preparing for (3)}\}$

$(\underline{A}y :: (\underline{A}b :: P.y.b) \Leftarrow (\underline{A}x : x<y : (\underline{A}a :: P.x.a)))$

$\Rightarrow \quad \{$ (3) with $Q.z := (\underline{A}b :: P.z.b)$, for $z = x, y$; and unnesting $\}$

$(\underline{A}y, b :: P.y.b)$.

<u>End</u> Proof of Theorem 0 .

<u>Proof of Theorem 1</u> We have to derive (3) and (4) from the validity of (2) for all P. Note that the theorem does not hold for lexical couplings with empty domain. Therefore, we expect the property

(5) lexical coupling has non-empty domain $\equiv$
component relations have non-empty domains

to come in handy.

<u>Proof of (3)</u> Starting at the antecedent, we get

$(\underline{A}y :: Q.y \Leftarrow (\underline{A}x : x<y : Q.x))$

$= \quad \{\text{non-empty ranges : (5)}\}$

$(\underline{A}y, b :: Q.y \Leftarrow (\underline{A}x, a : x<y : Q.x))$

$= \quad \{$ def. P: $Q.y \equiv P.y.c$, for all y and c $^{(*)}\}$

$(\underline{A}y, b :: P.y.b \Leftarrow (\underline{A}x, a : x<y : P.x.a))$

$\Rightarrow \quad \{$ anti-monotonicity, twice, using (1) : $x<y \Rightarrow (x,a) < (y,b)$ $\}$

antecedent of (2)

$\Rightarrow$

$\Rightarrow \quad \{(2)\}$

$(\underline{A}y,b :: P.y.b)$

$= \quad \{ (\underline{A}b :: P.y.b) \equiv Q.y$, from $(*)$ and $(5)\}$

$(\underline{A}y :: Q.y)$.

End Proof of (3) .

Proof of (4)

$(\underline{A}b :: R.b \Leftarrow (\underline{A}a : a<b : R.a))$

$= \quad \{$by (5) , $P \equiv (\underline{A}y :: P)$ and $P \equiv (\underline{A}x : x=y : P)$, for y and x respectively not in $P\}$

$(\underline{A}y,b :: R.b \Leftarrow (\underline{A}x : x=y : (\underline{A}a : a<b : R.a)))$

$= \quad \{$pred. calc.; and $R.c \equiv P.z.c$, for all z,c $(**)\}$

$(\underline{A}y,b :: P.y.b \Leftarrow (\underline{A}x,a : x=y \wedge a<b : P.x.a))$

$\Rightarrow \quad \{$by (1), $(x=y \wedge a<b) \Rightarrow (x,a) < (y,b)\}$

antecedent of (2)

$\Rightarrow \quad \{(2)\}$

$(\underline{A}y,b :: P.y.b)$

$= \quad \{ (\underline{A}y :: P.y.b) \equiv R.b$, by $(**)$ and $(5)\}$

$(\underline{A}b :: R.b)$

End Proof of (4) .

End Proof of Theorem 1 .

We are grateful to the Eindhoven Tuesday Afternoon Club for useful comments on an earlier version of this note; especially J.C.S.P. van der Woude's insistance that we obey our own rules and provide a heuristically convincing argument is appreciated.

Eindhoven, 18 September 1990

prof.dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
U.S.A.

dr. A.J.M. van Gasteren
Department of Mathematics and Computing Science
University of Groningen
P.O. Box 800
9700 AV Groningen
The Netherlands