# Interleaving is associative

For string $x$ and alphabets $A, B$, the projection operator $\upharpoonright$ has the properties

(0) $(x \upharpoonright A) \upharpoonright B = x \upharpoonright (A \cap B)$ ,

(1) $x \upharpoonright \phi = []$ ,

where [] stands for the empty string.

In what follows, alphabets $B, C, D$ and strings $b, c, d$ satisfy

(2) $B \cap C = \phi$ , $B \cap D = \phi$ , $C \cap D = \phi$
(3) $b \upharpoonright B = b$ , $c \upharpoonright C = c$ , $d \upharpoonright D = d$ .

<u>Remark</u> I wrote (3) rather than
$b \in B^*$ , $c \in C^*$ , $d \in D^*$
because strings of infinite length are admitted. (End of Remark)

<u>Exercise</u> Prove, for instance, $b \upharpoonright C = []$ . (End of Exercise.)

For a pair of strings from disjoint alphabets we define a nondeterministic operation of interleaving: "$y$ is an interleaving of $b$ and $c$" means that $y$ satisfies

(4) $y \upharpoonright B = b \wedge y \upharpoonright C = c \wedge y \upharpoonright (B \cup C) = y$

Since $(B \cup C) \cap D = \phi$ — this follows from (2) —, the last term of (4) tells us that we can now introduce $z$ as an interleaving of $y$ and $d$, i.e. $z$ satisfies

(5) $\quad z\restriction(B\cup C)=y \;\wedge\; z\restriction D=d \;\wedge\; z\restriction(B\cup C\cup D)=z$ .

In order to show that interleaving is associative we demonstrate

(i) that $z$ as constrained by (4) and (5) satisfies

(6) $\quad z\restriction B=b \;\wedge\; z\restriction C=c \;\wedge\; z\restriction D=d \;\wedge\; z\restriction(B\cup C\cup D)=z$

which is symmetric in the pairs $(b,B)$, $(c,C)$, and $(d,D)$.

(ii) that for $z$ as constrained by (6), a $y$ exists such that (4) and (5) are satisfied.

<u>Demonstration (i)</u>

We deal with the conjuncts of (6) in turn. For the first one we observe

$$
\begin{array}{ll}
 & b \\
= & \{(4),\ \text{leading conjunct}\} \\
 & y\restriction B \\
= & \{(5),\ \text{leading conjunct}\} \\
 & (z\restriction(B\cup C))\restriction B \\
= & \{(0)\ \text{and set theory}\} \\
 & z\restriction(B\cap B\ \cup\ C\cap B) \\
= & \{(2)\ \text{and set theory}\} \\
 & z\restriction B
\end{array}
$$

The next conjunct of (6) follows similarly, and the remaining two follow directly from (5). (End of Demonstration (i).)

Demonstration (ii)

The last 2 conjuncts of (5) follow directly from (6); in order to satisfy the leading conjunct we choose for $y$

(7) $\quad y = z \restriction (B \cup C)$

(Actually, we had little choice!). Our remaining obligation is to show that with this choice of $y$, (4) is satisfied. We deal with its conjuncts in turn; for the leading one we observe

$$
\begin{array}{ll}
 & y \restriction B \\
= & \{ * \cdots * ; \text{ (7) is (5)'s leading conjunct} \} \\
 & z \restriction B \\
= & \{ \text{(6), leading conjunct} \} \\
 & b \quad .
\end{array}
$$

The next conjunct of (4) follows similarly, and for the last conjunct we observe

$$
\begin{array}{ll}
 & y \restriction (B \cup C) \\
= & \{ (7) \} \\
 & (z \restriction (B \cup C)) \restriction (B \cup C) \\
= & \{ \text{(0) and set theory} \} \\
 & z \restriction (B \cup C) \\
= & \{ (7) \} \\
 & y \quad .
\end{array}
$$

(End of Demonstration (ii).)

The above demonstrations are totally trivial, and that is precisely the point. In my first effort I started with a recursive definition of being an interleaving, something like

$$
\begin{array}{l}
\text{interleaving}.[\,].y.z \;\equiv\; y=z \\
\text{interleaving}.x.[\,].z \;\equiv\; x=z \\
\text{interleaving}.(p{:}x).(q{:}y).(r{:}z) \;\equiv \\
\qquad (p=r \;\wedge\; \text{interleaving}.x.(q{:}y).z) \;\vee \\
\qquad (q=r \;\wedge\; \text{interleaving}.(p{:}x).y.z) \qquad ,
\end{array}
$$

which then invites an argument by mathematical induction. A grandiose mistake, for the whole problem has nothing to do with strings, and everything with projection, of which only (0) is used. (Property (1) is only needed for the exercise. It refers to the empty <u>string</u>; property (0) leaves the type of x pretty open.)

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