Equilateral triangles and rectangular grids

The other day, I encountered the following problem: "Does there exist an equilateral triangle whose vertices have integer (orthogonal Cartesian) coordinates?". The problem is, indeed, as elementary as it looks, but in solving it, I had a few pleasant surprises; hence this note. I urge the reader to think about the problem before reading on.

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My first concern was how to characterize the notion "equilateral triangle": all edges of the same length, all angles of the same size, or a mixture of the two. Considering that, in analytical geometry, lengths are more readily expressed than angles, I chose the first characterization, i.e. upon the triangle with vertices $(0,0)$, (a,b), (c,d) I imposed the requirement that its sides be of equal length. More precisely, my interest turned to integer quintuples (a,b,c,d,k) satisfying $(0) \wedge (1) \wedge (2)$ with

(0) $\quad a^2 + b^2 = k$

(1) $\quad c^2 + d^2 = k$

(2) $\quad (a-c)^2 + (b-d)^2 = k$.

Using (0) and (1), (2) can be simplified to

(2') $\quad 2\cdot(a\cdot c + b\cdot d) = k$.

Seeing that k is even, we turn our attention in view of (0) and (1) to squares and factors of 2. And here we have to make a little jump. It does not suffice to observe (the correct) $\text{even}.x^2 \equiv \text{even}.x$, which for a sum of 2 squares only implies $\text{even}.(x^2+y^2) \equiv \text{even}.x \equiv \text{even}.y$. We have to reduce squares modulo 4 and use

(3a) $\quad x^2 \,\underline{\text{mod}}\, 4 = 0 \;\equiv\; \text{even}.x$
(3b) $\quad x^2 \,\underline{\text{mod}}\, 4 = 1 \;\equiv\; \text{odd}.x$

from which we conclude about the sum of squares

(4a) $\quad (x^2+y^2) \,\underline{\text{mod}}\, 4 = 0 \;\equiv\; \text{even}.x \wedge \text{even}.y$
(4b) $\quad (x^2+y^2) \,\underline{\text{mod}}\, 4 = 1 \;\equiv\; \text{even}.x \not\equiv \text{even}.y$
(4c) $\quad (x^2+y^2) \,\underline{\text{mod}}\, 4 = 2 \;\equiv\; \text{odd}.x \wedge \text{odd}.y$.

And now we observe for an integer solution (a,b,c,d,k) of $(0) \wedge (1) \wedge (2')$:

$\quad k \,\underline{\text{mod}}\, 4 \neq 0$
$=\quad$ { (2'), in particular even.k }
$\quad k \,\underline{\text{mod}}\, 4 = 2$
$=\quad$ { (0) and (1) }
$\quad (a^2+b^2) \,\underline{\text{mod}}\, 4 = 2 \;\wedge\; (c^2+d^2) \,\underline{\text{mod}}\, 4 = 2$
$=\quad$ { (4c) twice }
$\quad \text{odd}.a \wedge \text{odd}.b \wedge \text{odd}.c \wedge \text{odd}.d$

$\Rightarrow$ {arithmetic}
even.$(a \cdot c + b \cdot d)$
$=$ {(2')}
$k \ \underline{\text{mod}}\ 4 = 0$.

Having thus proved

(5) $k \ \underline{\text{mod}}\ 4 = 0$,

we conclude on account of (0), (1), (4a)

(6) even.$a \wedge$ even.$b \wedge$ even.$c \wedge$ even.d .

If (a, b, c, d, k) is an integer solution of $(0) \wedge (1) \wedge (2)$, $(a/2, b/2, c/2, d/2, k/4)$ obviously solves that equation as well; (5) and (6) tell us now that the latter is again an <u>integer</u> solution. By mathematical induction we conclude that any solution of $(0) \wedge (1) \wedge (2)$ is divisible by any power of 2 (4), and hence equal to $(0, 0, 0, 0, 0)$, 0 being the only integer with an unbounded number of divisors. All zeros is indeed a solution, i.e. there exist equilateral triangles whose vertices have integer coordinates, but such a triangle is degenerate and its vertices coincide.

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For several reasons I was pleased with the above argument. Of course I regret the rabbit embodied by (3a)(3b), but I console myself with the thought that it is a fairly small one. The proof of $k \underline{mod} 4 = 0$ is a nice example of proving P by constructing a weakening chain from $\neg P$ to P, and the final part of the argument nicely exploits that only 0 has an unbounded number of divisors.

<u>Remark</u> The latter observation can be turned into a heuristic: a good way of showing that a Diophantine equation has no other solution than 0, it suffices to show that any solution p can be divided by some n $(n > 1)$ such that p/n is again a solution. (End of Remark)

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