Another ping-pong argument from Leibniz's principle

Here is —see EWD1186— another very simple proof in which our obligation to use Leibniz's principle leads to a ping-pong argument. The proof comes from the beginning of lattice theory.

Of infix operators ↑ ("up") and ↓ ("down") and x and y, we are given

(0) $x \uparrow (y \downarrow x) = x$

(1) $y \downarrow (x \uparrow y) = y$

The reader is more or less supposed to recognize the Absorption Laws. Please note that (0) and (1) are transformed into each other by the interchange $x, \uparrow \leftrightarrow y, \downarrow$.

We are asked to prove

(2) $x \uparrow y = x \quad \equiv \quad y \downarrow x = y$.

Please note that the two sides of this equivalence are transformed into each other by that same interchange $x, \uparrow \leftrightarrow y, \downarrow$.

Ping —i.e. $x \uparrow y = x \Rightarrow y \downarrow x = y$ — is proved by observing

$$
\begin{array}{ll}
 & y \downarrow x \\
= & \quad \{\text{LHS, i.e. } x \uparrow y = x\} \\
 & y \downarrow (x \uparrow y) \\
= & \quad \{(1)\} \\
 & y \qquad ,
\end{array}
$$

and pong now follows from the symmetries observed.

The ping-pong argument is forced upon us since (0) and (1) do not provide the tools for a (boolean!) value-preserving transformation of $x \uparrow y = x$ into $y \downarrow x = y$. We have to use the semantics of these two equality signs, i.e. we have to apply Leibniz's principle.

Austin, 15 October 1994

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