The theorem of de Ceva once more

(See also EWD989, where the theorem was erroneously attributed to Gupta.)

We identify the row vector $p = (p0, p1, p2)$ with the line p whose equation is $p0 \cdot x + p1 \cdot y + p2 = 0$, and similarly for q and r. We identify the column vector $P = (Px, Py, 1)$ with the point P, whose x,y-coordinates are (Px, Py), and similarly for Q and R. When the equations are normalized, a scalar product like $q \cdot R$ equals R's perpendicular distance from q.

We now consider lines p, q, r and points P, Q, R, such that P lies on p, Q lies on q and r goes through R, i.e.

(0) $p \cdot P = 0$, $q \cdot Q = 0$, $r \cdot R = 0$,

and now observe

(1) $(p \cdot Q) \cdot (q \cdot R) \cdot (r \cdot P) + (p \cdot R) \cdot (q \cdot P) \cdot (r \cdot Q) = 0$

$\equiv$ {definition of determinant}

$$\det \begin{vmatrix} 0 & p \cdot Q & p \cdot R \\ q \cdot P & 0 & q \cdot R \\ r \cdot P & r \cdot Q & 0 \end{vmatrix} = 0$$

$\equiv$ { (0) }

$$\det \begin{vmatrix} p\cdot P & p\cdot Q & p\cdot R \\ q\cdot P & q\cdot Q & q\cdot R \\ r\cdot P & r\cdot Q & r\cdot R \end{vmatrix} = 0$$

$\equiv$ { definition of product of matrices }

$$\det \left(\begin{vmatrix} p \\ q \\ r \end{vmatrix} \times \begin{vmatrix} P & Q & R \end{vmatrix} \right) = 0$$

$\equiv$ { determinant of product is product of determinants; product is zero means a factor is zero }

$$\det \begin{vmatrix} p \\ q \\ r \end{vmatrix} = 0 \quad \vee \quad \det \begin{vmatrix} P\, Q\, R \end{vmatrix} = 0$$

$\equiv$ { geometrical interpretation }

(lines p, q, r are concurrent or parallel)

$\vee$

(points P, Q, R are collinear) .

The usual formulation of the theorem of de Ceva states in the context of a nondegenerate triangle PQR (in our terminology)

(lines p, q, r are concurrent) $\Rightarrow$

$$\left(\frac{p\cdot Q}{p\cdot R}\right)\cdot\left(\frac{q\cdot R}{q\cdot P}\right)\cdot\left(\frac{r\cdot P}{r\cdot Q}\right) = -1$$

but

- this formulation is only an implication and not an equivalence

- both formulation and proof break down if some factors in (1) equal zero
- the quotient $p{\cdot}Q / p{\cdot}R$ is formulated as $\overline{AQ}/\overline{AR}$, where A is the point of intersection of p and QR; the case that p and QR are parallel is thus ruled out, and would require a separate proof.
- the case that lines p, q, r are parallel is traditionally ignored (also when the converse theorem is derived that concludes concurrency from (1)).

In short: three cheers for calculation!

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA