On the transitive closure of a wellfounded relation

by
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Dedicated to
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0 Notation, terminology, and imported mathematics

We use the term "predicates" for the elements of a boolean algebra, their standard model being boolean functions defined on some space. Besides the usual boolean operators $(\neg, \wedge, \vee, \Rightarrow, \Leftarrow, \equiv)$, we use for predicates the "everywhere operator" $[...]$ of type predicates$\rightarrow$ {<u>true</u>, <u>false</u>} : $[p]$ is only <u>true</u> if p is, for all other p, $[p] \equiv$ <u>false</u>. In the standard model, $[...]$ corresponds to universal quantification over the underlying space.

In what follows, f is a so-called "predicate transformer", i.e. a function f: predicates$\rightarrow$predicates. When we then wish to consider the boolean expression $[f.x]$ as an equation to be solved for the unknown x, we denote that equation by $x: [f.x]$. If

$$x: ([f.x] \wedge \langle \forall y: [f.y]: [x \Rightarrow y] \rangle)$$

has a solution, it is unique and is called "the strongest solution of $x: [f.x]$"; similarly, if

$$x: ([f.x] \wedge \langle \forall y: [f.y]: [x \Leftarrow y] \rangle)$$

has a solution, it is unique and is called "the weakest solution of $x: [f.x]$".

A predicate transformer f being "monotonic" means that for all predicates x, y

$$[x \Rightarrow y] \Rightarrow [f.x \Rightarrow f.y] \quad .$$

The famous Theorem of Knaster-Tarski tells us that for monotonic f, the equations $x: [f.x \Rightarrow x]$ and $x: [f.x \equiv x]$ have the same strongest solution

$$\langle \forall y: [f.y \Rightarrow y]: y \rangle \quad ,$$

while the equations $x: [x \Rightarrow f.x]$ and $x: [x \equiv f.x]$ have the same weakest solution

$$\langle \exists y: [y \Rightarrow f.y]: y \rangle \quad .$$

The "relation calculus" is obtained from the predicate calculus by adding to it two operators ("composition" and "converse") and one constant (the identity element of composition). In this note we only need the composition, which will be denoted by an infix ";" with a syntactic binding power between $\neg$ and the logical infix operators. We shall use that composition is associative and that in either argument it distributes over $\vee$; the latter property implies that composition is monotonic in either argument. In the standard model of the relation calculus, the predicates are boolean functions of two variables of the same type —the underlying space has the structure of a "Cartesian square"— and the composition of the relations p and q is modelled by

$$\alpha(p;q)\beta \;\equiv\; \langle \exists\gamma :: \alpha p \gamma \wedge \gamma q \beta \rangle \quad .$$

(As is usually done, we have treated the relations as infix symbols.)

Relation $\sqsupset$ being (left-)wellfounded means no more and no less than that for any P

$$\langle \forall\alpha :: \langle \forall\gamma : \alpha \sqsupset \gamma : P.\gamma \rangle \Rightarrow P.\alpha \rangle \;\Rightarrow\; \langle \forall\alpha :: P.\alpha \rangle \;,$$

i.e. that proof by mathematical induction is valid: instead of showing $P.\alpha$ for arbitrary α , it suffices to show that it follows from the assumption that P holds for all "smaller" values γ . Taking a contrapositive, denoting $\neg P$ by x (a relation that need not depend on its second argument), and $\sqsupset$ by r , we can express the (left-)wellfoundedness of r by the fact that for all x

(0) $\quad [x \Rightarrow r;x] \Rightarrow [\neg x]$.

In other words, (left-)wellfoundedness of r means that *false* is the only, and hence the weakest solution of $x: [x \Rightarrow r;x]$. But then, thanks to Knaster-Tarski, *false* is also the weakest, and hence the only, solution of $x: [x \equiv r;x]$, so that we can also characterize the (left-)wellfoundedness of r by the fact that for all x

(1) $\quad [x \equiv r;x] \Rightarrow [\neg x]$.

The two alternative formulations for r being (left-)wellfounded explain our interest in the boolean expressions $[x \Rightarrow r;x]$ and $[x \equiv r;x]$.

Remark Other characterizations of wellfoundedness are in terms of minimal elements of subsets or lengths of decreasing chains. In view of their explicit reference to the elements of the domain, the existence of the point-free characterizations (0) and (1) comes as a surprising simplification. (End of Remark.)

1 The nonreflexive transitive closure

The nonreflexive transitive closure s of a relation r can be defined as the strongest s satisfying

(2) $\quad [r \vee r;s \equiv s]$.

(Because of Knaster-Tarski, there are alternative definitions, but we chose (2) because, all by itself, it has interesting consequences.)

<u>Lemma 0</u> For any pair r, s of relations satisfying (2) we have

$$\langle \forall x :: [x \Rightarrow r;x] \Rightarrow [x \Rightarrow s;x] \rangle \quad .$$

<u>Proof</u> We observe for any r, s satisfying (2) and any x

$$
\begin{array}{ll}
 & [x \Rightarrow s;x] \\
\equiv & \quad \{(2)\} \\
 & [x \Rightarrow (r \vee r;s);x] \\
\Leftarrow & \quad \{\text{pred. calc. and monotonicity of } ;\} \\
 & [x \Rightarrow r;x]
\end{array}
$$

(End of Proof.)

Combining Lemma 0 with the characterization (0) of left-wellfoundedness, we conclude

<u>Corollary 0</u> If the nonreflexive transitive closure of a relation is left-wellfounded,

so is the relation itself.

<u>Lemma 1</u> For any pair r, s of relations satisfying (2) we have

$$\langle \forall x :: [x \equiv s;x] \Rightarrow [x \equiv r;x] \rangle$$

<u>Proof</u> We observe for any r, s satisfying (2) and any x satisfying

(3) $\quad [x \equiv s;x]$

$$\begin{array}{ll} & x \\ \equiv & \{(3)\} \\ & s;x \\ \equiv & \{(2)\} \\ & (r \vee r;s);x \\ \equiv & \{;\text{ distributes over } \vee \text{ and is associative}\} \\ & r;x \vee r;s;x \\ \equiv & \{(3)\} \\ & r;x \vee r;x \\ \equiv & \{\text{pred. calc.}\} \\ & r;x \end{array}$$

(End of Proof.)

Combining Lemma 1 with the characterization (1) of left-wellfoundedness, we conclude

<u>Corollary</u> 1 If a relation is left-wellfounded,

so is its nonreflexive transitive closure.

<u>Remark</u> It is worth noting that neither of our Lemmata requires r to be wellfounded or s to be the strongest s satisfying (2). (End of Remark.)

2 Uniqueness

Finally, we show that for left-wellfounded r, (2) determines s uniquely, i.e. given

(4) $[r \lor r;s \equiv s]$

(5) $[r \lor r;t \equiv t]$

(6) $\langle \forall x :: [x \Rightarrow r;x] \Rightarrow [\neg x] \rangle$,

then $[s \equiv t]$ holds.

<u>Proof</u> For reasons of symmetry, it suffices to show $[t \Rightarrow s]$. We observe

$$
\begin{array}{ll}
 & [t \Rightarrow s] \\
\equiv & \quad \{\text{pred. calc.}\} \\
 & [\neg (t \land \neg s)] \\
\Leftarrow & \quad \{(6) \text{ with } x := t \land \neg s\} \\
 & [t \land \neg s \Rightarrow r;(t \land \neg s)] \\
\equiv & \quad \{\text{shunting}\} \\
 & [t \Rightarrow s \lor r;(t \land \neg s)] \\
\equiv & \quad \{(4)\} \\
 & [t \Rightarrow r \lor r;s \lor r;(t \land \neg s)]
\end{array}
$$

$\equiv$ { ; distributes over $\vee$ and pred. calc }

$[t \Rightarrow r \vee r;(t \vee s)$

$\Leftarrow$ { monotonicity of ; }

$[t \Rightarrow r \vee r;t]$

$\equiv$ { (5) }

true

(End of Proof.)

I gratefully acknowledge the contribution of Rutger M. Dijkstra (viz. the isolation of Lemma 1) and that of Wim Feijen and Netty van Gasteren (viz. the final proof of unicity).

Austin, 10 January 2000