Formal study of plane Delaunay triangulation

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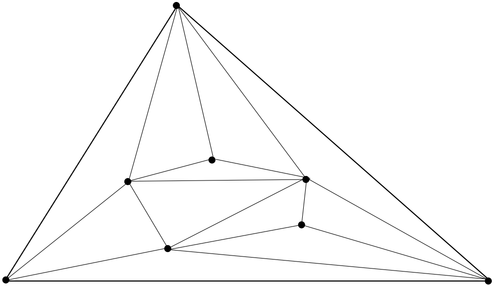
Introduction

- divide objects on the plane into triangles
- Choose a good data-structure for subdivisions
- Remove bad triangles: the ones that are too flat
- Delaunay criterion to recognize bad triangles
- Naive algorithm successive flips
- Method to guarantee termination

The algorithm at work



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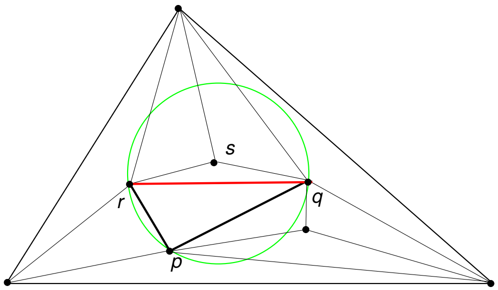

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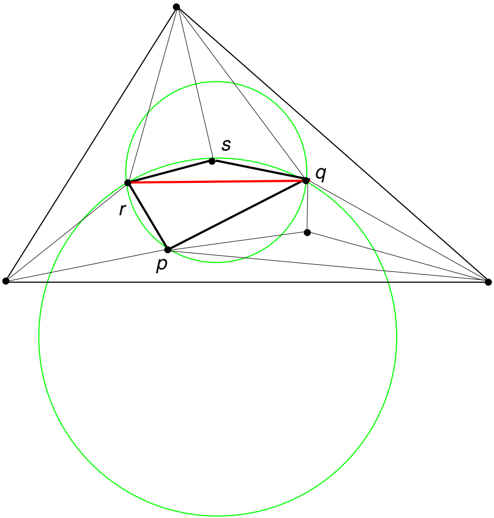

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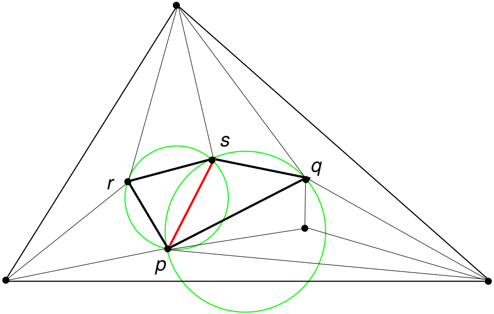

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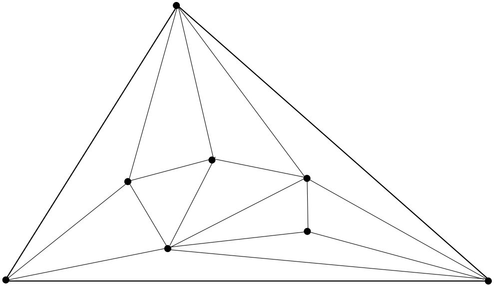

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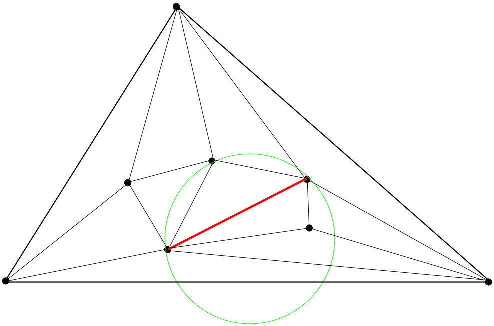

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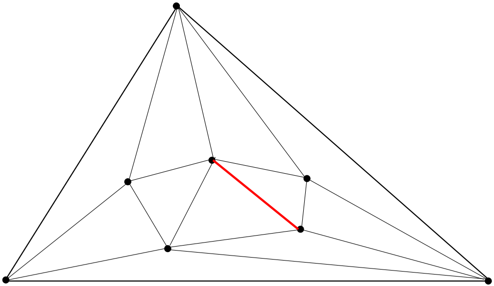

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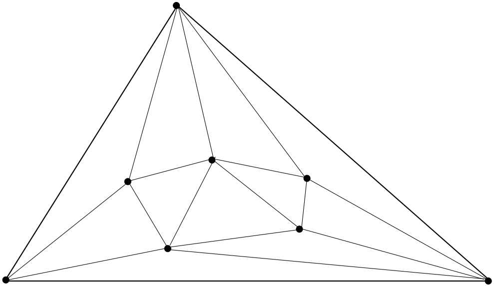


Hypermaps

- What elementary objects compose a triangulation?
- Idea : darts (half-edges),
- Connect darts together : two permutations α_0 α_1
- Two darts linked by α_0 constitute an edge (α_0 involutive)
- all darts in the same α_1 orbit constitute a point
- Implementation content: permutations represent pointers!

Two levels

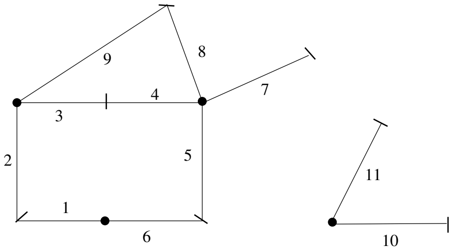
- Use a set of dart identfiers (typically `nat`)
- Use a list structure as a *free* map containg
  - darts
  - links between darts
- Add predicates for:
  - when darts can be added (only if not already present)
  - when links can be added (permutation properties, geometry constraints)
- Recognize faces: actually orbits for $\alpha_1^{-1} \circ \alpha_0^{-1}$

Characterizing triangulations

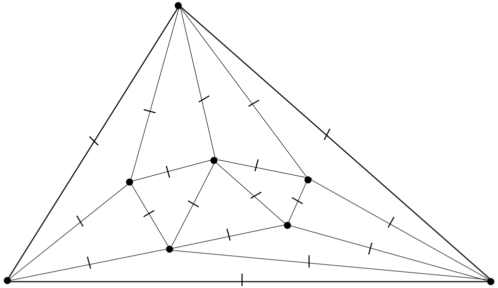
- Planar graphs: Euler formula on point, edge, and face counts
- Triangulations: all faces must be triangular
- Geometry: all triangles must be oriented
  - Except the outer face

Geometric presentation of subdivisions

D	1	2	3	4	5	6	7	8	9	10	11
α0	2	1	4	3	6	5	7	9	8	10	11
α1	6	3	9	5	7	1	8	4	2	11	10



Triangulations as hypermaps



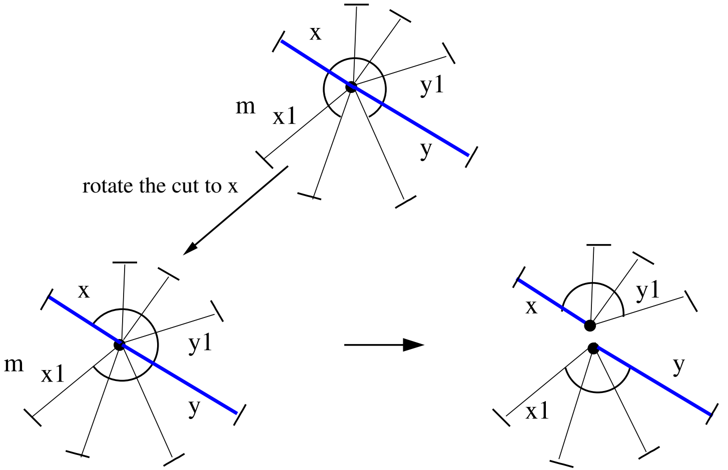
Fine points of permutation handling

- Invariant for map data-structure: only simple links
- $d' = \alpha_i(d)$,
  - if there is a link at dimension i from d to d'
  - if there is no link to d', no link from d, and a chain of links from d' to d
- Orbits are open, they have a cut
- The cut can be "rotated" in the orbit

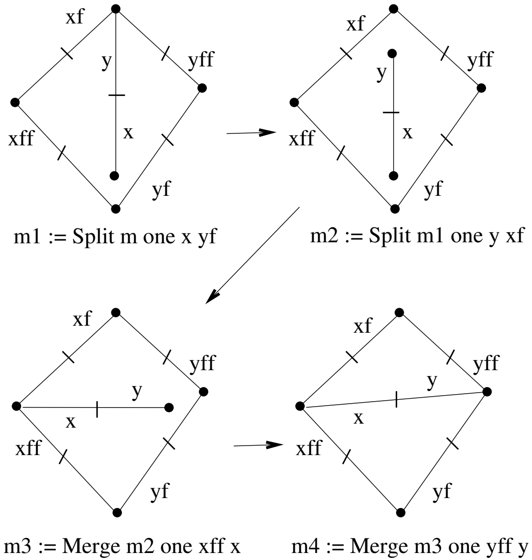
Splitting and merging

- Detaching objects: splitting point orbits
  - Keep together adjacent edges in two sets
  - Done by first rotating, then removing a link
- Merging points
  - Need rotating the orbits to choose how orbits arrange
  - Then add only one link
- Proofs required: make sure the hypermap invariants are preserved
- Merging points also requires a change of coordinates

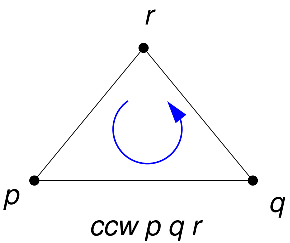
Splitting illustration



Flip : two splits, two merges



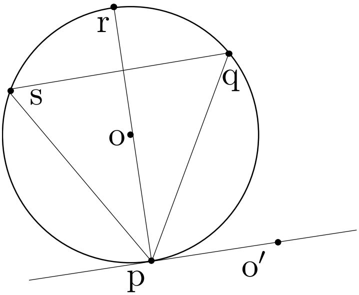
Orientation



Property expressed by an algebraic computation

$$\begin{vmatrix} x_p & y_p & 1 \\ x_q & y_q & 1 \\ x_r & y_r & 1 \end{vmatrix} > 0$$

Orientation and Flip



o is the center of the circumcircle

$p\ o\ q\ r\ s$ are in the configuration of a property known by Knuth

That property was proved formally in 2001 for convex hulls

Detecting illegal edges

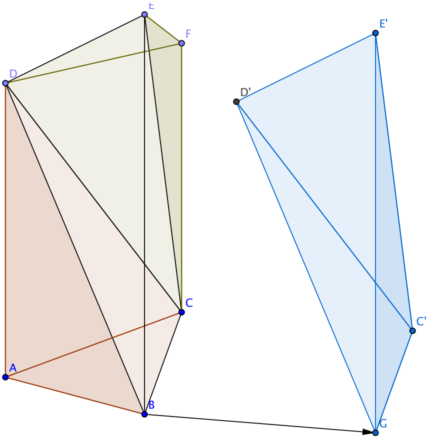
$$\begin{vmatrix} x_p & y_p & {x_p}^2 + {y_p}^2 & 1 \\ x_q & y_q & {x_q}^2 + {y_q}^2 & 1 \\ x_r & y_r & {x_r}^2 + {y_r}^2 & 1 \\ x_s & y_s & {x_s}^2 + {y_s}^2 & 1 \end{vmatrix} > 0$$

This determinant actually is a volume

Ensuring termination

- Project each point onto the revolution paraboloid $z = x^2 + y^2$
- Thus define a triangulated surface in space
- Consider the volume under this surface
- This volume decreases everytime one flips an illegal edge
- Termination argument: only a finite set of possible triangulations
- We developed a generic approach to describe this “finiteness” argument

Computing volumes



Conclusion

- Data structures closer to graphs
- Hypermaps actually represent efficient representations in memory
- Also adapted to handle more dimensions (add α_2 etcetera)
- Need to add a function to create the initial triangulation
- Need to also consider arbitrary external faces
- Wish: use this functional model as a basis to study an imperative implementation