Mapping among the nodes of infinite trees: A variation of Kőnig's infinity lemma

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Abstract. Koenig's infinity lemma states that an infinite rooted tree in which every node has finite degree has an infinite path. A variation of this lemma about mappings from one tree to another is presented in this note. Its proof utilizes Koenig's lemma, and Koenig's lemma follows from this variation.

Keywords: Koenig's infinity lemma; Mappings among trees.

1. Introduction

The infinity lemma of Kőnig [1] has many applications in computer science, particularly in the study of program semantics in dealing with infinite computations. Typically, a single computation is represented by a sequence of events, called a *trace*, and an infinite computation by an infinite trace. Nondeterministic programs give rise to trees where each path in the tree is a possible trace. Denotational semantics often requires mapping such trees to other trees. A variation of Kőnig's lemma is useful in this context when both trees are infinite [2]. The rest of this note is concerned with a generalization of Kőnig's lemma; understanding of program semantics is not required to follow the note.

2. Variation of Infinity Lemma

Let S and T be two rooted trees whose nodes we also designate by S and T. Let *cover* be a binary relation over $S \times T$. Say that x covers y (y is covered by x) if $(x, y) \in cover$. For sets of nodes X and Y in S and T, respectively, X covers Y (Y is covered by X) if each node of Y is covered by a node of X. Observe that a node of S may cover zero, finite or infinite number of nodes. Further, nodes of S and T may have infinite degree. The theorem below gives conditions under which every path of T is covered by some path of S.

Theorem. Given S, T and *cover* as above, suppose:

- C1. Each node of T is covered by a non-empty finite set of nodes of S.
- C2. If node x covers node y then the ancestors of x in S (that includes x) cover the ancestors of y.

Then every path of T is covered by some path of S.

Proof of the Theorem. Without loss in generality, add a new root s to S, t to T and the pair (s, t) to *cover*. The hypotheses and the conclusion are unaffected by this construction.

The theorem is of interest only when T is infinite, because for finite T every terminal node y of T is covered by some node of S whose ancestors cover the path to y, from (C2).

Let p be a path in T; without loss in generality assume that p starts at root t and p is infinite. We show that p is covered by some path of S. Construct a tree R from S and p, as follows. The nodes of R are $\{(x, y) \mid (x, y) \in \mathit{cover},\ y \in p\}$. Clearly, (s, t) is a node in R. The parent of every other node (x', y') of R is (x, y), where y is the parent of y' in p and x the closest ancestor of x' in S that covers y. Such an x exists because of condition (C2). Node x may possibly be x'. Then R is tree with root (s, t). Observe:

1. Every node of p is the second component of a distinct node of R. Hence, if p is infinite so is R.
2. Every node of R has finite degree: node (x, y) of R has children of the form (x', y') where y' is the unique child of y in p. From (C1), y' is covered by a finite number of nodes, so (x, y) has finite degree.
3. Apply Kőnig's lemma in conjunction with items (1) and (2) to establish the existence of an infinite path q in R. Let q_1 and q_2 be the sequences of the first and second components, respectively, of q. By construction, $q_2 = p$. And q_1 corresponds to a path of S because (x, y) is the parent of (x', y') in q where x is an ancestor of x' in S. The path corresponding to q_1 covers p. This concludes the proof. (Notice that the sequence q_1 can include some nodes multiple times, and any one node z infinitely often; in such a case the path corresponding to q_1 is finite and terminates at z.)

3. A Proof of Kőnig's Lemma

Kőnig's infinity lemma is easily established from this theorem. Given an infinite tree with finite degree at each node, we have to show the existence of an infinite path. Let S be the given tree and T just a path whose nodes are numbered consecutively from the root with natural numbers. Let x cover n where n is the level of x in S. Condition (C1) is met because every level of S has a finite non-zero number of nodes, and (C2) is easily seen to be met. So, there is a path of S that covers T, and since each node of a path of S has a different level, the path is infinite.

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[1] D. Kőnig. *Theorie der Endlichen und Unendlichen Graphen: Kombinatorische Topologie der Streckenkomplexe*. Leipzig: Akad. Verlag, 1936.

[2] J. Misra. A denotational semantic theory of concurrent systems. Available at `http://www.cs.utexas.edu/users/misra/DenotationalSemantics.pdf`.