On the union of well-founded relations: An application of Koenig's Lemma
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Problem: We are given two well-founded relations, R, S, on the same set, and that $(R \cup S)$ is transitive. We show that $(R \cup S)$ is well-founded.

Solution: We take an arbitrary chain in $(R \cup S)$ and show that it is finite in length. The proof strategy is to construct a tree out of the elements of the given chain so that (1) sons of each node are elements of a chain in S, and (2) each path in the tree corresponds to a chain in R. Since R, S are well-founded, it follows that each node has finite degree and all paths are finite. Applying Koenig's lemma, the tree is finite, and therefore, the chain is finite.

To simplify matters, we add a new value T to the set and postulate that yRT for all y in the set. Consider any chain, C, in $(R \cup S)$ for which we will prove finiteness. We may assume that the the first element of C is T since T may be added to the front of any chain to form a chain. Note that the elements of C need not have distinct values though we will treat elements at different positions of C as distinct.

Proposition 1: For elements u, v in C where u precedes v, $(vRu) \vee (vSu)$

Proof: Since u, v are elements of a chain in $(R \cup S)$ and u precedes v, from the transitivity of $(R \cup S)$

$v \ (R \cup S) \ u$

i.e., $(vRu) \vee (vSu)$ []

Next, we construct a tree from C as follows. Each element of C is a node in the tree; T is the root. For an element v, $v \neq T$, its *father* is u where u is the closest element preceding v for which (vRu). Each element v has a father because the first element, T , satisfies vRT. The father relation defines a tree because the father of v precedes v in C, for all v, $v \neq T$.

Proposition 2: Let u, v be siblings in the tree and u precede v in C. Then, (vSu).

Proof: Let w be the common father of u, v. From the definition of father, w is the closest element preceding v such that vRw. Since u precedes v and w precedes u, u is closer to v than w. Therefore, $\neg(vRu)$, otherwise, u would have been v's father. Combining this result with $(vRu) \vee (vSu)$ – which follows from Proposition 1 since u precedes v – we conclude (vSu). []

Theorem: The chain C is finite.

Proof: We show that the tree is finite, as sketched earlier in the paper. First, we show that each node in the tree has finite degree. Take any node and arrange its sons in a sequence, d, in the order in which they appear in C. For any two consecutive elements, u, v, in d, u precedes v in C, and, hence, (vSu), from Proposition 2. Therefore, d is a chain in S; from the well-foundedness S, d is finite.

Next, every path in the tree is a chain in R, because for a node v with father u, vRu. From the well-foundedness of R, every path in the tree is finite.

Koenig's Lemma states that a tree is finite if and only if every node has finite degree and every path is finite. Therefore, our tree is finite, and so is the chain C. []

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