Coloring Grid Points, without Rabbits and Snakes
Jayadev Misra
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Problem: The following verbatim description is from Dijkstra, EWD-1248.

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Show that, for any finite set of grid points in the plane, we can
colour each of the points either red or blue such that on each grid
line the number of red points differs by at most 1 from the number of
blue points.
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This problem can be reduced to a simple problem on graphs. Construct a graph in which each grid line is a vertex, and there is an edge between two vertices when the corresponding grid lines share a specified grid point. It is required to color the edges of this graph – which is finite, since there are a finite number of grid points – using two colors so that the *polarity* of each vetrtex is at most 1. Here, polarity of a vertex is the absolute value of the difference between the number of red and blue edges incident on it.

Observation: The graph is bipartite.
Proof: A vertex corresponds to either a vertical or a horizontal grid line. Each edge connects a "vertical" vertex to a "horizontal" vertex. □

Each cycle in a bipartite graph has an even number of edges. Therefore, the edges of a cycle may be colored alternately, red and blue, and such a coloring does not affect the polarity of any vertex. Hence, we may remove the cycles from the graph (in arbitrary order) and solve the coloring problem over the remaining edges.

After removing the cycles we are left with an acyclic undirected graph, i.e., a *tree* (or, possibly, a *forest*). The different trees in a forest may be independently colored, as follows. If a tree has zero edges, the coloring is trivial. For a tree with nonzero number of edges, there is an edge (u, v) which is the only edge incident on vertex v. Color the edges other than (u, v) inductively. Then color (u, v) to meet to polarity constraint: if u has more red edges incident on it than blue ones, color (u, v) blue, otherwise red.