**A Counting Problem Communicated by Dijkstra**

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**Problem:** Given is a 10 X 10 square grid. A cell is a square in the grid. Two cells are neighbors if they share a side. Initially, some 9 cells are colored red. A *step* colors a cell red provided it has at least two red neighbors. Prove that no matter which 9 cells are initially red, not all cells can be colored red (by repeated application of the step).

**Solution:** Call each side of a cell an *edge*. An edge is *incident* on a cell if it forms one of its sides.

**Observe**: There are 4 edges incident on each cell, and there are 40 edges that form the boundary of the region. An interior edge (i.e., one not on the boundary) is incident on exctly 2 cells. Each boundary edge is incident on one cell only.

Call an edge *open* if it is incident on exactly one red cell. Initially there are at most 36 open edges, because there are 9 red cells with at most 4 open edges per cell. A step does not increase the number of open edges, because a step "closes" at least two open edges and creates at most 2 open edges. Therefore, the number of open edges remains at most 36. If all cells are reddened then all boundary edges will be open, and thus, there will be 40 open edges. Therefore, not all cells can be reddened.

Thanks: E.W.Dijkstra communicated this problem to me.