Knowledge, Product and Sum

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There are two individuals, *Product* and *Sum*, who are given integers p, s, respectively. Here, $p = m \times n$ and $s = m + n$, for some unknown integers m, n, $2 \leq m < 100$, $2 \leq n < 100$. It is required to deduce the values of m, n given the following exchange.

Product:: I don't know m, n.
Sum :: I knew that.
Product:: Now I know them.
Sum :: Now I know them too.

The following solution is a slightly simplified version of a solution originally due to Dijkstra (EWD-666).

1. First, Product says "I don't know them". We deduce that not both m, n are primes (otherwise, Product would have been able to deduce their values by factoring p). Further, if a prime factor of p is 50 or more then Product can claim that this factor is one of the unknowns, because otherwise if this is a factor of m, say, then m exceeds 99. So

   $P1(m, n)$:: (composite $m \vee$ composite n) $\wedge$ each prime factor of m and n is less than 50.

2. Next, sum says "I knew that". It means
   $(\forall u, v : u + v = s : P1(u, v))$
   Otherwise, if there is a pair $(u, v), u+v = s$, that violates P1, it is possible that $(m, n) = (u, v)$; then Product would have been able to predict (m, n). In particular, s is not a sum of two primes. For small values, every even number exceeding 2 is a sum of two odd primes (Goldbach's Conjecture). Therefore
   s is odd, i.e., one of the unknowns is odd and the other even.
   Also, from $P1(2, s - 2)$,
   $s - 2$ is composite.
   Now, if $s > 54$, then the pair $(53, s - 53)$ violates the second conjunct of P1. Therefore
   $s \leq 54$.

We enumerate the possible values of s by considering every composite odd integer below 52, and adding 2 to each such number.

$$s \in S \text{ where } S = \{11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53\}. \qquad (0)$$

Henceforth, whenever we write a pair (u, v), we take u to be even, v to be odd, and both are assumed to be between 2 and 99.

3. Next, Product says "Now I know them". A pair (u, v) is *valid* if Product can deduce that $m, n = u, v$ given that $p = uv$. We observe:

   - For odd prime q, $(2^i, q)$ is valid. (1)
   - For any x, $(32, x)$ is valid. (2)
     Define (x, y) to be a *co-pair* of (u, v) if the two pairs are distinct and $uv = xy$.
   - If every co-pair (x, y) of (u, v) has $x + y \notin S$ then (u, v) is valid. (3)

   Since one of m, n is even and the other odd the only feasible factorization of $(2^i \times q)$ is into $m, n = 2^i, q$, justifying (1). Rule (2) follows from the fact that the only factorization of $32 \times x$ is into $m, n = 32, x$ because m contains 32 as a factor, and m can't be any larger because $m + n \leq 54$. Rule (3) says that any co-pair whose components sum to a value outside S, is not a possible value for (m, n).

4. Next, Sum says "Now I know them too". This means that there exists a unique pair (u, v) such that $(s = u + v) \wedge (u, v)$ is valid. Therefore, all members of S that can be written as $u + v$ and $x + y$ where (u, v) and (x, y) are both valid cannot be s. In the following table, we eliminate all but one member, 17, from S by demonstrating valid (u, v) and (x, y); in each case (u, v) is valid from Rule 1, the reason (x, y) is valid is given in the table. The only case that needs some explanation is for 29, where (4,25) is claimed to be valid. The only co-pair of (4,25) is (20,5) and $20 + 5 = 25 \notin S$. Therefore,(4,25) is valid using Rule (3).

Element	(u, v)	(x, y)	Why (x, y) is valid
11	4,7	8,3	Rule 1
17	4,13		
23	4,19	16,7	Rule 1
27	4,23	8,19	Rule 1
29	16,13	4,25	See Description
35	4,31	32,3	Rule 2
37	8,29	32,5	Rule 2
41	4,37	32,9	Rule 2
47	4,43	32,15	Rule 2
51	4,47	32,19	Rule 2
53	16,37	32,21	Rule 2

   For the remaining value, 17, $17 = 4 + 13$ where (4,13) is valid. This shows that $m, n = 4, 13$ is a solution. We prove uniqueness of the solution by enumerating all pairs x, y, where $x + y = 17$ (excepting the pair (4,13)), and showing that (x, y) is not valid. We prove (x, y) is not valid using Rule (3); i.e., by demonstrating a co-pair (u, v) in each case where $u + v \in S$.

x, y where $17 = x + y$	u, v where $xy = uv$	$u + v (\in S)$
2,15	6,5	11
3,14	2,21	23
5,12	20,3	23
6,11	2,33	35
7,10	2,35	37
8,9	24,3	27

Thus, 4,13 is the unique solution.