Fibonacci Numbers

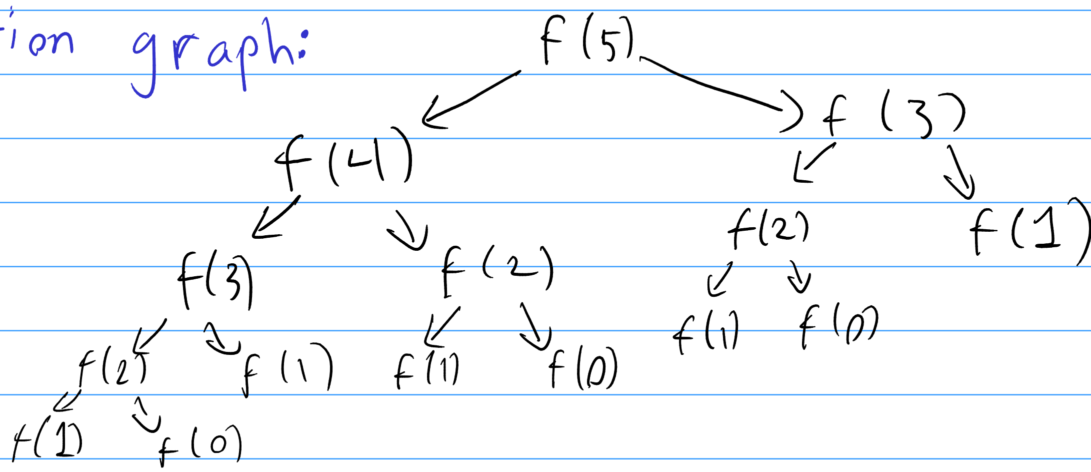
$$F_0 = 0, \quad F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}.$$

How quickly can we compute the n^{th} Fibonacci number?

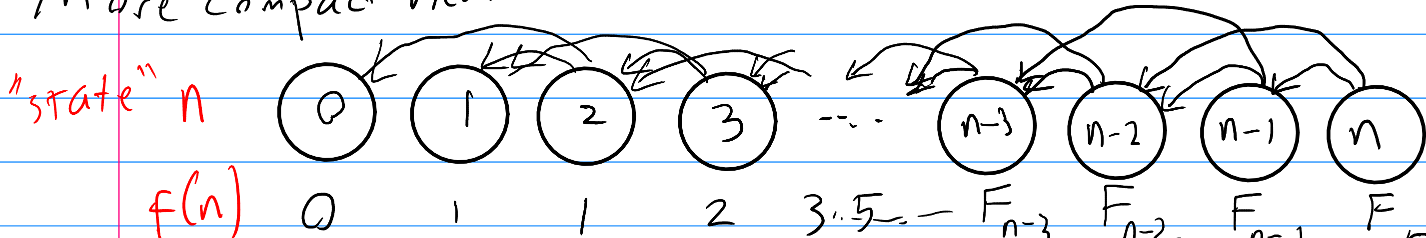
Naive recursion:

```
def    f(n):
    if n ≤ 1: return n
    return  f(n-1) + f(n-2)
```

Computation graph:



More compact view:



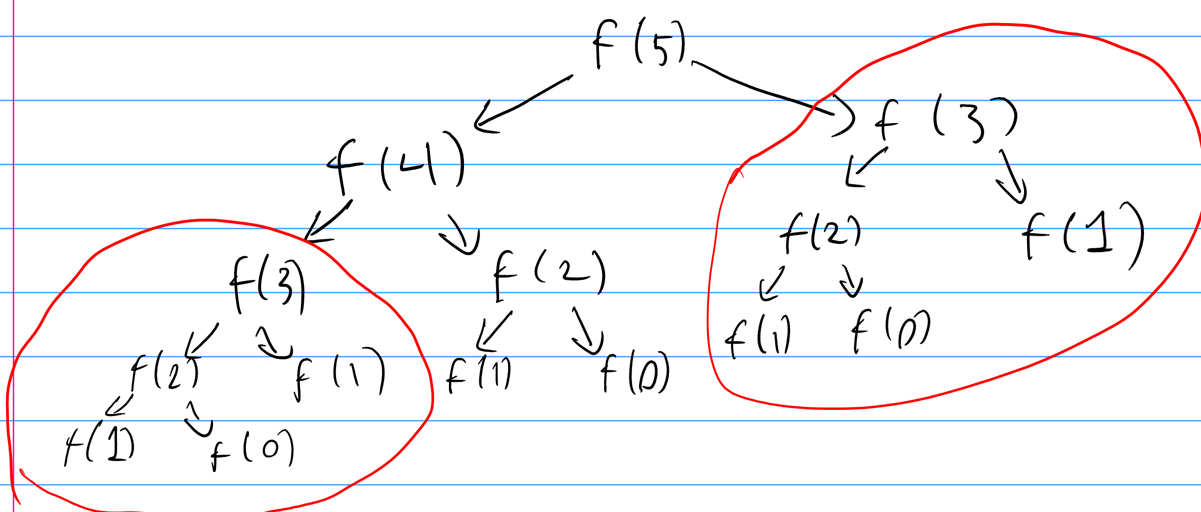
Returns to f(1) again and again

$$T(n) = T(n-1) + T(n-2) + \boxed{\text{Something} \geq 1}$$

$$\Rightarrow T(n) \geq \underset{(\approx 1.6^n)}{F_n} > 2^{n/2}$$

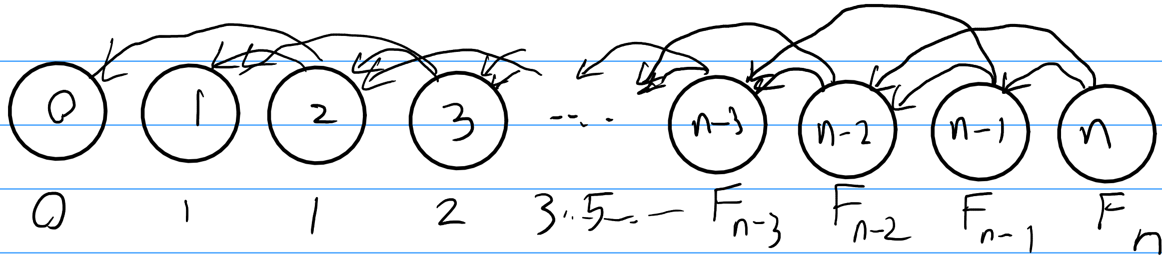
How can we avoid this repetition?

Only compute each F_i once.



f(3) never changes. Can store the answer.

in this view: Compute node (= "state" = "input") once.



Dynamic Programming (DP)

"top-down"
memoization: Keep a record of each answer you compute.
Before recursion, check the memo.

"bottom-up" DP: compute answers low -> high.

```
def fib(n):
    fibs = [0, 1]
    for i in range(2, n+1):
        fibs.append( fibs[i-1] + fibs[i-2])
    return fibs[n]
```

Time: $O(n)$
Space: $O(n)$

"sliding window" DP:

```
def fib(n):
    a, b = 0, 1
    for i in range(2, n+1):
        a, b = b, a+b
    return b
```

Time: $O(n)$
space: $O(1)$

Matrix Exponentiation Method.

Can view each step of the sliding window method as a matrix multiply.

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

The final $\begin{bmatrix} a \\ b \end{bmatrix}$ is thus $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \ldots$$

F_n = bottom right of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1}$.

How quickly can we compute A^n for a 2×2 matrix A?

Repeated Squaring:

First compute $A,\ A^2,\ A^4 = (A^2)^2,\ A^8 = (A^4)^2, \ldots$

up to A^{2^k} for $k = \lfloor \log_2 n \rfloor$

Express n in base 2:

$$n = \sum_{i=0}^{k} a_i \cdot 2^i$$

Then $$A^n = A^{\left(\sum_{i=0}^{k} a_i 2^i\right)} = \prod_{i=0}^{k} A^{a_i \cdot 2^i} = \prod_{i : a_i = 1} A^{2^i}$$

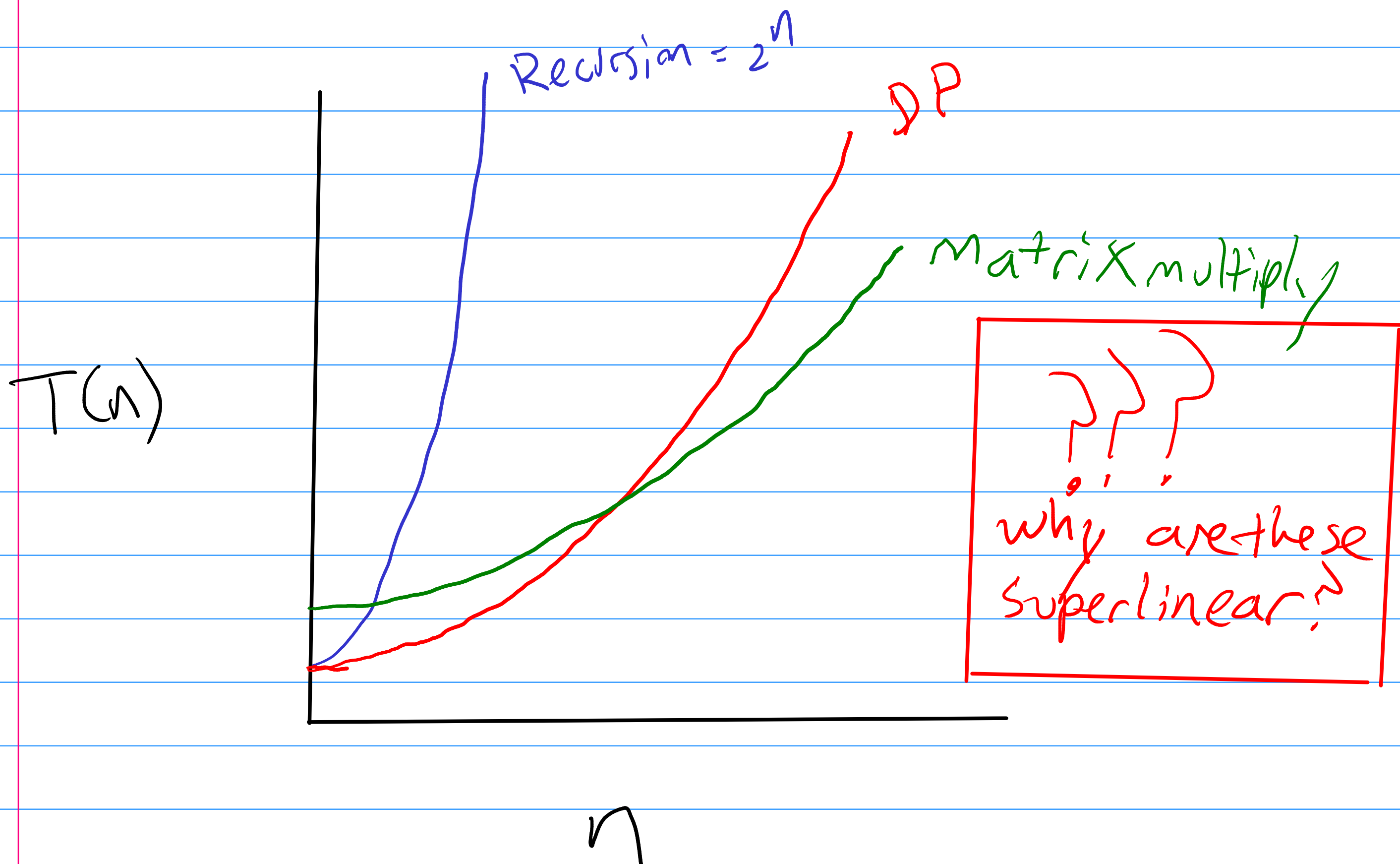
Example:

Suppose $n = \underbrace{101101_2}_{k \text{ bits}} = 45$

Then $A^{45} = A^{32} \cdot A^{8} \cdot A^{4} \cdot A$

Total time = $O(\log n)$ 2x2 matrix multiplications
= $O(\log n)$.

Easy to implement these algorithms, benchmark time.



All the preceding time/space bounds are wrong

Because the numbers get big.

$F_n \approx 1.6^n$ takes $\Theta(n)$ bits to store.

DP: $\sum_{i=1}^{n} n = \boxed{\Theta(n^2) \text{ time}}$

Matrix Multiplication:

the $A^{2^{k+1}} = (A^{2^k})^2$ step involves multiplying $\Theta(2^k)$-bit numbers.

Then Fibonacci has

$T(n) = T(\frac{n}{2}) + (\text{multiplication time for } n \text{ bits})$

$= \Theta(\text{multiplication time for } n \text{ bits})$

(if this is $> n^{1.01}$)

Elementary: $\Theta(n^2)$

Karatsuba: $\Theta(n^{\log_2 3})$

For the rest of this course, avoid this issue

- only deal with numbers bounded by $n^{O(1)}$
- Assume computer can manipulate such numbers in $O(1)$ time

[64-bit word size, $n < 2^{64}$]