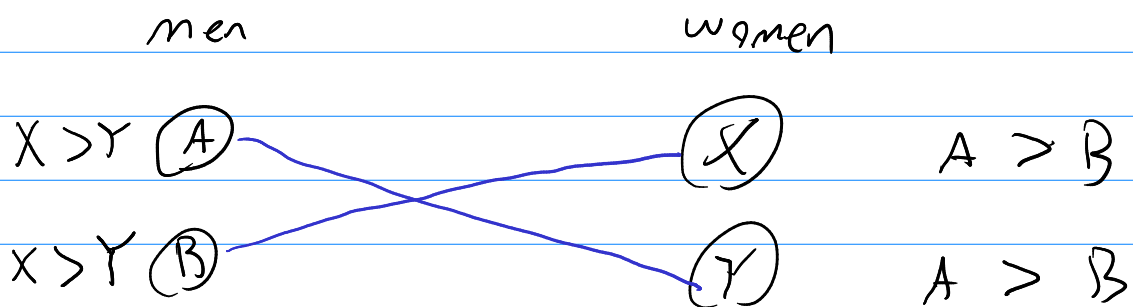
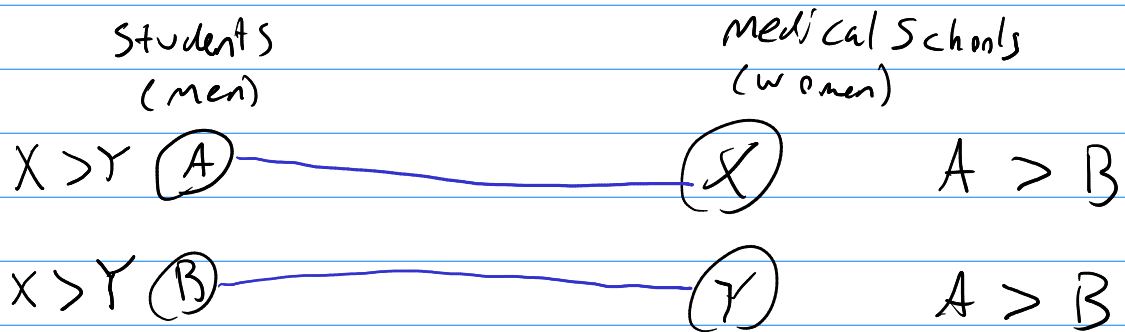


Stable Matching

(a.k.a. Stable Marriage)

Match n students
to n positions

Each student/position has ranked preferences

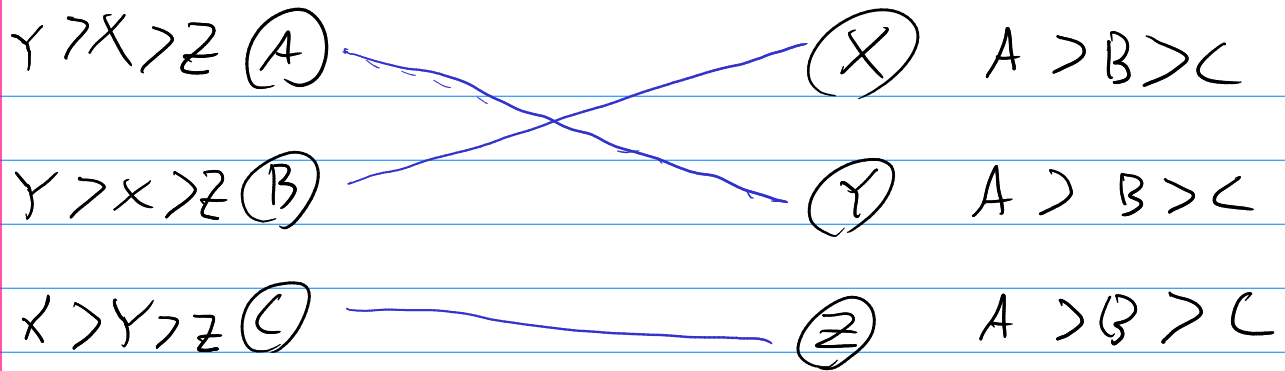


↑ unstable A & X will elope

Matching is stable if
no unmatched pair both prefer
each other to current partners.

Goal: given all preference lists, find a
stable matching.

Example



Gale-Shapley:

Start with nobody matched.

Repeat:

Inf. nite:

ACB $Y > X > Z$ (AX, BY, CZ)
CAB $Z > X > Y$ (AY
ABC $X > Y > Z$ $\rightarrow CY$
 $\rightarrow CX$
 $\rightarrow AX$)

Pick any unattached man.

This man proposes to next woman on his list.

The woman:

if unattached or preferred to

current match, **accepts** & breaks up ^{with current match}

else, she prefers current match \Rightarrow **reject**

if rejected, man crosses her off his list.

Claims

(1) This terminates.

- every woman's happiness never decreases.

- in each round, either:

(1) name crossed off a man's list

(2) woman gets happier by ≥ 1

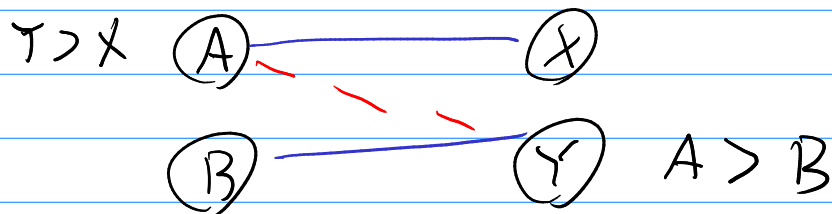
n men & n per list \Rightarrow only n^2 times

n women & n per list \rightarrow only n^2 times

$\Rightarrow O(n^2)$ steps

(2) The final state is stable

suppose otherwise, Then in final state
Some A would elope with Y, while
A matched to X & Y to B.



How did this happen?

For A to be matched to X,
A proposed to X

\Rightarrow A at some point proposed to Y.

\Rightarrow at some point, Y had a mate
she thought \geq A in quality

But Y's happiness never decreases

\Rightarrow Y's final happiness is \geq her view of A,
which B is not, $\Rightarrow \Leftarrow$.

\Rightarrow Gale-Shapley gives a stable matching in $O(n^2)$ time.

Theorem: Gale-Shapley is optimal for men
& pessimal for women!

[Of all stable matchings, A gets best match in any
X worst in any]

Lemma: each man only rejected by women that
 cannot be matched ^{to him} in any stable matching
 (or broken up with)

Proof (induct on # rounds)

in a given round, suppose X rejects A
 in favor of B .

B has been rejected by all he prefers to X
 \Rightarrow all of those infeasible for B , by induction

\Rightarrow in any stable matching,
 B would elope with X if X agrees.

\Rightarrow in any stable matching, X is not matched to A
 (would cause elopement)

$\Rightarrow A$ infeasible for X .

in $G-S$, man gets favorite woman that never rejects him
 \Rightarrow optimal for man, by lemma.

Thus: if Gale-Shapley matches man A to woman X ,

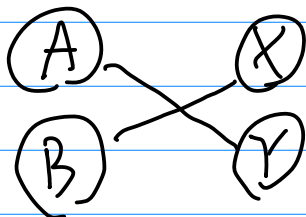
$X = \text{best}(A) =$ best possible match in any
 stable marriage

finally, want: $A = \text{worst}(X)$

consider any stable matching where A & X not
 matched; say $A-Y$, $X-B$.

① $G-S$
 is optimal
 for men

$X > Y$



②

$B > A$

because else
 A & X would elope

$\Rightarrow A$ is worse than any alternative
 B that X can be stably matched
 to