Linear Programming

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CS 331H

Class Outline

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  - Trucks carry twice as much as cars.

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- Mathematically:

$$\begin{aligned} \text{maximize:} \quad & C + 2T \\ \text{subject to:} \quad & 2C + 3T \leq 12 \\ & C + 5T \leq 15 \\ & C, T \geq 0 \end{aligned}$$

Solving small cases by hand

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  - Find all vertices, and for each:
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- Algebraically:
  - Find all vertices, and for each:
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- Geometrically:
  - Draw the picture of all feasible points
  - Slide in the direction of the objective until you get stuck.

General Linear Programming (LP)

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Optimize (maximize or minimize) a *linear objective* in many variables, subject to *linear constraints* on them $(=, \leq, \geq)$.

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\begin{aligned}
\text{maximize:} \quad & x_1 + 3x_2 - 345x_3 + x_4 \\
\text{subject to:} \quad & x_1 - 17x_2 \leq x_4 + 12 \\
& x_4 - x_3 \geq x_2 \\
& 67x_2 - 3x_1 = 83 \\
& x_3 \leq 0
\end{aligned}
$$

Formulations of LP

Standard form (or "symmetric")

For m constraints on n variables, given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$:

$$\begin{aligned} \text{maximize:} \quad & c \cdot x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

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Common alternative forms

"Alternative form"

$$\begin{aligned} \text{maximize:} \quad & c \cdot x \\ \text{subject to:} \quad & Ax \leq b \end{aligned}$$

or

"Slack form"

$$\begin{aligned} \text{maximize:} \quad & c \cdot x \\ \text{subject to:} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

The forms are reducible to each other

Standard	Alternative	Slack
$\max c \cdot x$	$\max c \cdot x$	$\max c \cdot x$
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  - Best theoretical result: $O(n^{2.38} L)$ time (Cohen, Lee, Song '19).

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  - Number of iterations depends on problem instance & rule for choosing next vertex, but could be exponential.

Finding an initial feasible vertex

- Simplex works, eventually, once you have a feasible vertex.
- Doesn't seem so useful:

Problem

If you can solve "does this polytope have any feasible point" you can also solve linear programming (= optimize over polytopes).

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Proof.

We want to determine $OPT = \max c \cdot x$ s.t. $Ax \leq b$. Then $OPT \geq \tau$ if and only if the polytope

$$Ax \leq b$$
$$c \cdot x \geq \tau$$

has any solution. So if we can solve this, we binary search on τ to solve LP. □

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Finding a feasible point

We want to find a point x such that $Ax \leq b, x \geq 0$. Introduce a new variable $z \in \mathbb{R}$, and solve:

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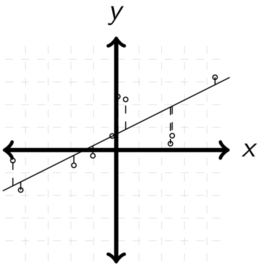
- NEW has $n + 1$ variables, one tight constraint of the optimum is $z \geq 0$, and the other n are among $Ax \leq b, x \geq 0$.
- Hence the solution $\widehat{x}$ is a vertex of the original LP.

Class Outline

L1 linear regression



Given n points on plane: $(x_1, y_1), \ldots, (x_n, y_n)$. Find the line $mx + b$ minimizing the average error:

$$\text{Err} = \frac{1}{n} \sum_{i=1}^{n} |y_i - (mx_i + b)|$$

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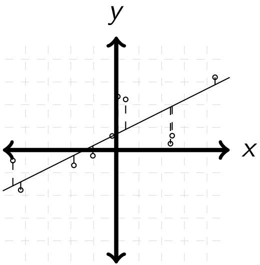

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Part (2): Now, minimize the *maximum* error.

Writing old problems as linear programs

- Write network flow as a linear program
- Write shortest paths as a linear program
- Write minimum cut as a linear program