CS 388R: Randomized Algorithms, Fall 2023 Oct 25th

Lecture 18: Spectral Sparsification of Graphs

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NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 Overview

In the last lecture we discussed the Bernstein Concentration Inequality and the Rudelson-Vershynin (RV) Lemma, which we are going to use in this lecture.

1.1 Rudelson-Vershynin (RV) Lemma

As presented as follows: $x_1, ..., x_m \in R^n$ and they are independent, $\forall i$, $\max_i \leq K$, $\|\mathbf{E}[x_i x_i^T]\| \leq 1$,

$$\mathbf{E}[\|\frac{1}{m}\sum_{i=1}^{m} x_i x_i^T - \frac{1}{m}\sum \mathbf{E}[x_i x_i^T]\|] \lesssim K\sqrt{\frac{\log n}{m}} \tag{1}$$

if $K\sqrt{\frac{\log n}{m}} \leq 1$

2 Problem Setup

Given a graph $G = (V, E)$, unweighted and un-directed graph with n vertices and m edges. The graph Laplacian G is defined by $n \times n$ matrix $L_G = D - A$,

- $A \in R^{n\times n}$: the graph adjacency matrix, $A(u, v) = 1, \forall$ *edge* $(u, v) \in E$, otherwise 0
- $D \in R^{n\times n}$ is the diagonal matrix of vertex degree, i.e. $D(u, u) = \sum_{v\in V} A(u, v)$

For the ease of our discussion, we define the vertex-edge injection matrix $U \in R^{m\times n}$ as follows, the direction doesn't matter since we are dealing with an undirected graph. Given an edge $e = (u, v)$

$$U_{i,j} = \begin{cases} 1 & \text{if i = e, j = u} \\ -1 & \text{if i = e, j = v} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We then denote u_e^T as the row of U corresponding to e, then the Laplacian can be rewritten into

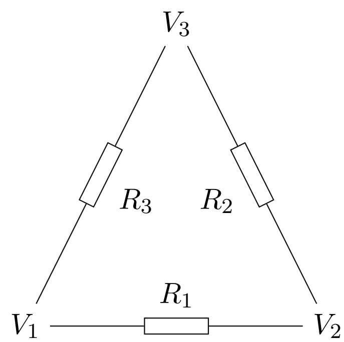
$$L_G = U^T U = \sum_{e\in E} \mathbf{u}_e \mathbf{u}_e^T \tag{3}$$

Generalize it to a weight graph $H = (V, \tilde{E}, W)$, W is a $R^{\|\tilde{E}\| \times \|\tilde{E}\|}$ diagonal weight matrix

$$L_H = U^T W U = \sum_{e \in \tilde{E}} \mathbf{w}_e \mathbf{u}_e \mathbf{u}_e^T \tag{4}$$

3 Physical intuition

An example of a resistor network G as below, with resistors$R_1 = R_2 = R_3 = 1\ \Omega$, $\frac{1}{w_e}$ resistance and enforce voltages V_1, V_2, V_3 on three points



3.1 Case 1: Fixed Voltages

Given voltages $\mathbf{v} \in R^n$, current across edge $e = (i, j)$ is

$$\mathbf{v}_i - \mathbf{v}_j = \mathbf{u}_e^T \mathbf{v} \tag{5}$$

$\overrightarrow{i} = U\mathbf{v}$ is current on every edge, where $U \in R^{m \times n}$. Assuming $\mathbf{r} = \mathbf{1}$, the total power used can be written as:

$$Power = \mathbf{i}\mathbf{v} = \mathbf{i}^2\mathbf{r} = \mathbf{i}^2 = \mathbf{v}^T U^T U \mathbf{v} = \mathbf{v}^T \cdot L_G \cdot \mathbf{v} \tag{6}$$

3.2 Case 2: Fixed Currents

Given $\mathbf{y} \in R^m$, the currents across edges,

$$x = U^T \cdot \mathbf{y} \tag{7}$$

$$\text{Net current into v} = \mathbf{x}_v = \sum_{\text{e out of } v} \mathbf{y}_e - \sum_{\text{e into } v} \mathbf{y}_e \tag{8}$$

Given a power source 1A into i and 1A out of j,

$$x = U^T \mathbf{y} = U^T U \mathbf{v} = L_G \mathbf{v} \tag{9}$$

$$V = L_G^\dagger x + \lambda \cdot \mathbf{1} \tag{10}$$

The effective resistance

$$R_{eff} = U_{ij}^T \mathbf{v} = \mathbf{v}_i - \mathbf{v}_j = U_{ij}^T L_G^\dagger U_{ij} \tag{11}$$

4 Graph Sparsification

Goal: Given a (dense) graph G, find (sparse, weighted) graph H, s.t. $L_H \approx L_G$.

4.1 Spectral Sparsifier

$$\forall x,\ P_H(x) = (1 \pm \epsilon) P_G(x) \tag{12}$$

$$\Leftrightarrow \forall x,\ (1-\epsilon) \cdot x^T L_G x \leq x^T L_H x \leq (1+\epsilon) x^T L_G x \tag{13}$$

$$\Leftrightarrow (1-\epsilon) L_G \preceq L_H \preceq (1+\epsilon) L_G \tag{14}$$

4.2 Cut Sparsifier

For any s, $Cut_H(s) = \sum_{e \in H} w(e) \cdot \mathbf{1}_{\|e \cap s\|=1}$

$$Cut_H(s) = (1 \pm \epsilon) \cdot Cut_G(s) \tag{15}$$

Set $x = 1\ \in\ S, 0\ otherwise$ then $Cut_H(s) = x^T L_H x$

$$x^T L_H x = \sum_e w_e \cdot (u_e^T x)^2 = \sum_{e=(i,j)} w_e \cdot (x_i - x_j)^2 \tag{16}$$

4.3 Randomized Sparsification

Given a weight graph, $L_G = \sum w_e u_e u_e^T$,

Algorithm 1 Randomized algorithm

Require: some probability p_e for each edge's importance
  for i = 1, 2, ..., M **do**
    `pick` $e_i \sim E$ `proportional to` p_{e_i}
    add e_i to H with weight $\frac{w_{e_i}}{M p_{e_i}}$
  end for

In one round (M = 1),

$$Z_e = \begin{cases} \sqrt{\frac{w_e}{p_e}} \cdot u_e & if\ e\ is\ picked \\ 0 & otherwise \end{cases} \tag{17}$$

$L_H = \sum_e z_e z_e^T \Rightarrow E[L_H] = \sum_e p_e \cdot \frac{w_e}{p_e} u_e u_e^T = L_G$,
As $M \to \infty$, we can get L_H to match L_G, the question is "How fast?" and "What p_e we should pick?"

4.3.1 Warmup: Complete Graph

Suppose we have a complete graph of n vertices

$$L_G = n \cdot I_n - \mathbf{1}\mathbf{1}^T \tag{18}$$

$$x^T L_G x = x^T(nI - \mathbf{1}\mathbf{1}^T)x = n \cdot \|x\|^2 \quad \textit{if } x \perp \mathbf{1} \tag{19}$$

If H is a spectral sparsifier, then

$$\|L_H - L_G\| \leq n \cdot \epsilon \Rightarrow x^T L_H x \leq x^T L_G x + n \cdot \epsilon \|x\|^2 \tag{20}$$

$$\Rightarrow \sup \frac{x^T(L_H - L_G)x}{x^T L_G x} = \frac{x^T(L_H - L_G)x}{n \cdot \|x\|^2} \leq 1 + \epsilon \tag{21}$$

Now let's pick p_e as follows:

$$p_e = \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)} \tag{22}$$

Define Z_e as follows:

$$Z_e = \begin{cases} \sqrt{\frac{w_e}{p_e}} \cdot u_e & \textit{if } e \textit{ is picked} \\ 0 & \textit{otherwise} \end{cases} \tag{23}$$

And then set y_i for $i = 1, ..., M$

$$y_i \coloneqq \frac{1}{\sqrt{n}} \cdot \sqrt{\frac{w_e}{p_e}} \cdot u_{e_i} \tag{24}$$

As $y_i y_i^T = \frac{1}{n} \cdot \sum_e Z_e Z_e^T$,

$$\mathbf{E}[y_i y_i^T] = \frac{1}{n} \cdot L_G \tag{25}$$

$\|L_G\| = n$, because G is a complete graph,

$$\|E[y_i y_i^T]\| = \frac{\|L_G\|}{n} = 1 \tag{26}$$

$$\|y_i\| = \|u_{e_i}\| \cdot \sqrt{\frac{1}{n \cdot \frac{1}{\binom{n}{2}}}} = \sqrt{2} \cdot \sqrt{\frac{n-1}{2}} = \sqrt{n-1} < \sqrt{n} \tag{27}$$

$$\frac{1}{n} L_H = \frac{1}{M} \sum_{i=1}^{M} y_i y_i^T \tag{28}$$

$$\begin{aligned} \mathbf{E}[\|L_H - L_G\|] &= n \cdot \mathbf{E}[\|\frac{1}{n} L_H - \frac{1}{n} L_G\|] \\ &= n \cdot \mathbf{E}[\|\frac{1}{m} \sum y_i y_i^T - \mathbf{E}[\frac{1}{m} \sum y_i y_i^T]\|] \\ &\lesssim n \cdot \sqrt{\frac{n \log n}{m}} \\ &\leq \epsilon \cdot n \qquad\qquad \texttt{for } m > \frac{1}{\epsilon} \cdot n \log n \end{aligned} \tag{29}$$

4.3.2 Arbitrary Graph

Recall the previous notation, $L_G = U^T U$, where U is the vertex-edge injection matrix. Define projection matrix onto $span(u)$ (set of possible current induced by voltages), as

$$R = U L_G^T U^T \in R^{m \times m} \tag{30}$$

and because R is a projection matrix, we have $R^2 = R$
Then we define $S \in R^{m \times m}$, the diagonal sampling-and-reweighting matrix, sampling M times with replacement, each e with probability p_e

$$S_{ee} = \frac{\# \; times \; e \; sampled}{M \cdot p_e} \tag{31}$$

Notice that $\mathbf{E}[S] = I$,

$$L_H = U^T S U \Rightarrow \mathbf{E}[L_H] = L_G \tag{32}$$

Q: how to pick p_e? How many samples do we need, s.t,

$$(1-\epsilon) L_G \lesssim L_H \lesssim (1+\epsilon) L_G \Leftrightarrow \|RSR - R\| \leq \epsilon$$

, and $\mathbf{E}[RSR] = R$

Let $y_i = \frac{1}{\sqrt{p_{e_i}}} \cdot R_{e_i}$, where R_e is the e^{th} column of R then,

$$\begin{aligned} \frac{1}{M} \sum_{i=1}^{M} y_i y_i^T &= \sum_{i=1}^{M} R_{e_1} \cdot \frac{1}{M p_{e_i}} \cdot R_{e_i}^T \\ &= RSR \end{aligned} \tag{33}$$

Using the RV Lemma,

$$\mathbf{E}[\|RSR - R\|] \lesssim K \sqrt{\frac{\log(m)}{M}} \tag{34}$$

where $K = max \|y_i\|$, hence

$$K \leq \max_e \frac{\|R_e\|}{\sqrt{p_e}} = \max_e \sqrt{\frac{r_e}{p_e}} \tag{35}$$

where r_e is defined as,

$$r_e = \|R_e\|^2 = u_e^T L_G^\dagger L_G L_G^\dagger u_e = u_e^T L_G^\dagger u_e \tag{36}$$

We can minimize K by setting $p_e \propto r_e$

$$p_e = \frac{r_e}{\sum_e r_e} = \frac{r_e}{n-1} \tag{37}$$

Then we have $K \leq \sqrt{n-1}$