CS 388R: Randomized Algorithms, Fall 2023 Oct 30th

Lecture 19: Markov Chains I

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NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 Overview

In this lecture, we are going to talk about Markov Chains. A Markov chain is a (discrete) memoryless stochastic process. For example, a random walk on n states, the distribution of position at time $t+1$ depends only on time t, not further history is needed.

2 Math things

Defining the transition matrix P as follows,

$$P_{i,j} = Pr[X_{t+1} = j | X_t = i] \tag{1}$$

Let $q^{(t)}$ be the distribution of X_t,

$$q^{(t+1)} = q^{(t)} \cdot P \tag{2}$$

Π is a stationary distribution, if $\Pi \cdot P = \Pi$. Markov Chain is "ergodic" if:

- Π is unique
- $\forall q^{(0)}, q^{(t)} = q^{(0)} \cdot P^t$ as $t \leftarrow \infty$

2.1 Fundamental Theorem of Markov Chains

A chain is ergodic when:

- Finite n
- Aperiodic: $\forall$ states, $gcd(\textit{loops from that state}) = 1$
- Irreducible: $\exists\ i \to j$ path $\forall i, j$

h_{uv}: hitting time from u to v starting at u, $\mathbf{E}[\textit{time to reach } V]$
C_{uv}: commute time, expected time to go from u to v and then back to $u = h_{uv} + h_{vu}$

Random walks in undirected graphs

$$P_{uv} = \begin{cases} \frac{1}{d(u)} & if(u,v) \in E \\ 0 & otherwise \end{cases} \tag{3}$$

Connected, no-bipartite $\Rightarrow$ ergodic.

$$\Pi_v = \frac{d(v)}{2m} \tag{4}$$

Check: $\sum_v \Pi_v = \frac{\sum d(0)}{2m} = 1$,

$$(\Pi P)_v = \sum_{(u,v)\in E} \Pi_u \cdot \frac{1}{d(u)} = \sum_{(u,v)\in E} \frac{1}{2m} = \frac{d(v)}{2m} = \Pi_v \Rightarrow \Pi P = \Pi \Rightarrow \Pi \textit{ is stationary} \tag{5}$$

with $\Pi_i > 0 \ \forall i \ N(i,t) \coloneqq \#$ times reach state i before time t,

$$\lim_{t\to\infty} \frac{N(i,t)}{t} = \Pi_i \tag{6}$$

Hitting time $h_{ii} = \frac{1}{\Pi_i}$ and $h_{ii} = \mathbf{E}[\textit{time to return to } i \textit{ after leaving it}]$