

# Distributed Distance Sensitivity Oracles\*

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## Abstract

We present results for the distance sensitivity oracle (DSO) problem, where one needs to preprocess a given directed weighted graph  $G = (V, E)$  in order to answer queries about the shortest path distance from  $s$  to  $t$  in  $G$  that avoids edge  $e$ , for any  $s, t \in V, e \in E$ . No non-trivial results are known for DSO in the distributed CONGEST model even though it is of importance to maintain efficient communication under an edge failure.

Let  $n = |V|$ , and let  $D$  be the undirected diameter of  $G$ . Our first DSO algorithm optimizes query response rounds and can answer a batch of any  $k \geq 1$  queries in  $O(k + D)$  rounds after taking  $\tilde{O}(n^{3/2})$  rounds to preprocess  $G$ . Our second algorithm takes  $\tilde{O}(n)$  rounds for preprocessing, and then it can answer any batch of  $k \geq 1$  queries in  $\tilde{O}(k\sqrt{n} + D)$  rounds. We complement these algorithms with some unconditional CONGEST lower bounds that give trade-offs between preprocessing rounds and rounds needed to answer queries.

Additionally, we present almost-optimal upper and lower bounds for the related all pairs second simple shortest path (2-APSiSP) problem, where for all pairs of vertices  $x, y \in V$ , we need to compute the minimum weight of a simple  $x$ - $y$  path that differs from the precomputed  $x$ - $y$  shortest path by at least one edge.

## 1 Introduction

In distributed networks, maintaining communication in the event of a link failure is an important problem. In a communication network modeled by a graph  $G = (V, E)$ , consider the failure of some edge  $e \in E$ . We investigate the problem of computing a shortest path distance when such an edge fault occurs. This is a fundamental problem in networks for routing and communication. Specifically, we need to answer queries of the form  $d(s, t, e)$  for  $s, t \in V, e \in E$ , where  $d(s, t, e)$  is the shortest path distance from  $s$  to  $t$  when edge  $e$  is removed from  $G$ . In the special case when  $s$  and  $t$  are fixed vertices this is known as the replacement paths (RPaths) problem.

Let  $|V| = n, |E| = m$ . A naive algorithm could compute all  $n^2m$  distances  $d(s, t, e)$  (this can be reduced to  $O(n^3)$  distances since we are only interested in edges on the  $s$ - $t$  shortest path), but such an algorithm would have high complexity ( $\Omega(n^2)$  rounds in CONGEST) simply due to the output size. So, we instead consider Distance Sensitivity Oracles (DSO), which have been studied in sequential setting. In DSO we first preprocess the network  $G$  and store certain information about distances. Then, we answer query  $d(s, t, e)$  using the stored information. Without any preprocessing, we can answer a query using a single source shortest path (SSSP) computation. We present two algorithms for computing DSOs in the CONGEST model, one that optimizes query response round complexity and one that prioritizes preprocessing round complexity while answering queries faster than SSSP. We also present lower bounds.

In the sequential setting, the first algorithm for constructing DSO was given in [12], which was improved in [6] to obtain an algorithm with  $\tilde{O}(mn)$  preprocessing time and  $O(1)$  query time. Sequential DSOs have been further studied in [34, 18, 10, 19]. In the distributed setting, near-linear round upper and lower bounds for RPaths in the CONGEST model are given in [25] for directed weighted graphs.

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Our main results in this paper are for DSO, for which we present two algorithms with differing trade-offs between preprocessing and query rounds in the CONGEST model. We also present lower bounds. Another problem we consider is the closely related all pairs second simple shortest paths problem (2-APSiSP), where instead of answering queries  $d(s, t, e)$  when a particular edge  $e$  fails, we are only interested in a simple  $s$ - $t$  path that differs from the  $s$ - $t$  shortest in  $G$  by at least one edge. The shortest such path is called the second simple shortest path (2-SiSP), and we denote its distance by  $d_2(s, t)$ . In the 2-APSiSP problem, we need to compute  $d_2(s, t)$  for all pairs of vertices  $s, t \in V$ . In this paper we show that we can compute 2-APSiSP in  $\tilde{O}(n)$  rounds and we present a lower bound to show that this is near-optimal. Unlike the DSO problem, there is no need for separate preprocessing and query response rounds.

A core procedure used in sequential DSO and 2-APSiSP algorithms [12, 6, 1] is that of computing excluded shortest paths distances, where we need to compute shortest path distances when certain types of paths are avoided one at a time. We present a distributed CONGEST procedure for this computation which we use in our 2-APSiSP algorithm and our first DSO algorithm.

**Roadmap.** The rest of this paper is organized as follows. After presenting background information and problem definitions in Section 1.1 we describe our results in Section 1.2. We describe prior work in Section 1.3. In Section 2 we present our distributed algorithm for excluded shortest paths, a core computation we use in DSO and 2-APSiSP algorithms. We present our DSO algorithms and lower bounds in Section 3 and our near-optimal 2-APSiSP upper and lower bounds in Section 4.

## 1.1 Preliminaries

### 1.1.1 The CONGEST Model

In the CONGEST model [29], a communication network is represented by a graph  $G = (V, E)$  where nodes model processors and edges model bounded-bandwidth communication links between processors. Each node has a unique identifier in  $\{0, 1, \dots, n-1\}$  where  $n = |V|$ , and each node only knows the identifiers of itself and its neighbors in the network. Each node has infinite computational power. The nodes perform computation in synchronous rounds, where each node can send a message of up to  $\Theta(\log n)$  bits to each neighbor and can receive the messages sent to it by its neighbors. The complexity of an algorithm is measured by the number of rounds until the algorithm terminates.

We mainly consider directed weighted graphs  $G$  in this paper, where each edge has an integer weight known to the vertices incident to the edge. Following the convention for CONGEST algorithms [11, 26, 17, 3, 4], the communication links are always bi-directional and unweighted.

### 1.1.2 Notation and Terminology

We will be dealing primarily with directed weighted graphs  $G = (V, E)$ . Let  $|V| = n$ . Each edge  $(s, t) \in E$  (for  $s, t \in V$ ) can have non-negative integer weight  $w(s, t)$  according to a weight assignment function  $w : E \rightarrow \{0, 1, \dots, W\}$ , where  $W = \text{poly}(n)$ . We denote the shortest path distance from  $s$  to  $t$  by  $d(s, t)$ , and a shortest path from  $s$  to  $t$  by  $P_{st}$ . The undirected diameter of  $G$ , denoted  $D$ , is the maximum shortest path distance between any two vertices in the underlying undirected unweighted graph of  $G$ . We use  $G^r = (V, E^r)$  to denote the reversed graph of  $G$ , where each edge is oriented in the opposite direction.

We use  $d(s, t, e)$ , for  $s, t \in V, e \in E$ , to denote the shortest path distance from  $s$  to  $t$  in the graph  $G - \{e\}$ , i.e., the graph  $G$  with edge  $e$  removed, also known as the replacement path distance. We generalize this definition for a path  $P$ , and  $d(s, t, P)$  denotes the shortest path distance from  $s$  to  $t$  in  $G - P$ . We use  $d_2(s, t)$ , for  $s, t \in V$ , to denote the second simple shortest path distance from  $s$  to  $t$  (2-SiSP distance), which is the minimum distance of a simple  $s$ - $t$

path that differs from a precomputed  $s$ - $t$  shortest path  $P_{st}$  by at least one edge. Note that  $d_2(s, t) = \min_{e \in P_{st}} d(s, t, e)$ .

*Shortest path trees and independent paths:* We denote the out-shortest path tree rooted at a vertex  $x \in V$  as  $T_x$ . For a subpath  $P$  of  $T_x$ , we define  $T_x(P)$  to be the subtree of  $T_x$  rooted at the vertex in  $P$  furthest from source  $x$ . We define the notion of a set of *independent* paths  $\mathcal{R}$  with respect to a shortest path tree  $T_x$  as follows: Each path  $P \in \mathcal{R}$  is a subpath of  $T_x$ , and for any pair of paths  $P, P' \in \mathcal{R}$ , for any two vertices  $u \in P, u' \in P'$ , the subtree of  $T_x$  rooted at  $u$  and the subtree of  $T_x$  rooted at  $u'$  are disjoint.

### 1.1.3 CONGEST Primitives.

We state some basic CONGEST primitives used in our algorithms.

In an unweighted graph, a breadth-first search (BFS) from  $k$  source vertices up to  $h$  hops takes  $O(k+h)$  rounds [23, 20]. We can broadcast  $m$  units ( $O(\log n)$  bits each) of information to all other vertices in  $O(m + D)$  rounds [29]. Given a rooted tree  $T$  of depth  $t$  in the CONGEST network (each node in the tree knows its incident edges), we can send  $m$  units of information down from root to all nodes in  $T$  in  $O(m+t)$  rounds – this operation is called a downcast [29], and we can even perform an associative operation on the received values at each node before propagating the information. A similar upcast can be done from leaves to root, and a convergecast is an upcast along a  $D$ -depth tree containing all vertices, and takes  $O(m + D)$  rounds [29].

*Scheduling with random delays:* We often use scheduling with random delays in order to efficiently perform multiple computations on the network. Following [14, 22], consider  $k$  distributed procedures  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$  that are to be run together on the network. Define *dilation* to be the maximum round complexity of any  $\mathcal{A}_i$ . For any edge  $e$  in the network, *congestion*( $e$ ) is defined to be the total number of messages through  $e$  over all  $\mathcal{A}_i$ , over all rounds where messages are sent through  $e$ . Define *congestion* =  $\max_{e \in E} \text{congestion}(e)$ . Then, using independent random delays for each of the  $k$  procedures, we can run them in  $O(\text{congestion} + \text{dilation} \cdot \log n)$  rounds [14]. In our algorithms, we often schedule  $k$  instances of the same procedure with different inputs; in this case dilation would same as the round complexity of a single procedure, and congestion would be  $k$  times the congestion of a single procedure.

We use *SSSP* and *APSP* to denote the round complexity in the CONGEST model for weighted single source shortest paths (SSSP) and weighted all pairs shortest paths (APSP) respectively. The current best algorithm for weighted APSP runs in  $\tilde{O}(n)$  rounds, randomized [7]. For weighted SSSP, [9] provides an  $\tilde{O}(\sqrt{n} + n^{2/5+o(1)}D^{2/5} + D)$  round randomized algorithm. The current best lower bounds are  $\Omega\left(\frac{\sqrt{n}}{\log n} + D\right)$  for weighted SSSP [30, 33] and  $\Omega\left(\frac{n}{\log n}\right)$  for (weighted and unweighted) APSP [26]. In our algorithms, we also use a low congestion SSSP procedure presented in [16], which takes  $\tilde{O}(n)$  rounds but incurs only  $\tilde{O}(1)$  congestion per edge. When computing shortest path distances from a source  $x$ , we assume that each vertex  $v$  knows the preceding and succeeding vertices on a shortest path from  $x$  to  $v$ . This can be achieved using  $O(1)$  rounds per source, by each vertex sharing its distance from  $x$  with its neighbors, e.g.  $y$  precedes  $v$  on the  $x$ - $v$  shortest path if  $d(x, v) = d(x, y) + w(y, v)$ .

### 1.1.4 Problem Definitions.

We first define the problem of excluded shortest paths. We then define the DSO and 2-APSiSP problems in the CONGEST model. In the following definitions,  $G = (V, E)$  is a directed weighted graph.

*Excluded Shortest Paths:* Given a set of sources  $X \subseteq V$  and sets of independent paths  $\mathcal{R}_x$  for each  $x \in X$  (as defined in 1.1.2), we need to compute  $d(x, y, P)$  for each  $x \in X, y \in V$  and  $P \in \mathcal{R}_x$ , i.e., the  $x$ - $y$  shortest path distance when path  $P$  is removed. The input set of paths  $\mathcal{R}_x$  is given as follows: Each edge  $e \in E$  knows whether or not it is part of some path  $P \in \mathcal{R}_x$ ,

and the identity of  $P$  if it is. For the output, each distance  $d(x, y, P)$  must be known at node  $y$ . *Distance Sensitivity Oracles (DSO)*: We need to compute  $d(x, y, e)$ , the  $x$ - $y$  shortest path distance when  $e$  is removed, for any pair of vertices  $x, y \in V$  and  $e \in E$ . A DSO algorithm is allowed to perform some preprocessing on the graph  $G$ , and then answer queries of the form  $d(x, y, e)$ . We assume that the query  $d(x, y, e)$  is known to every vertex in the graph. The round complexities of our algorithms are unchanged if we instead assume the query is known to only one node, since we can broadcast the query in  $D$  rounds once it is known. After the query is answered, distance  $d(x, y, e)$  must be known at node  $y$ .

*All Pairs Second Simple Shortest Path (2-APSiSP)*: We need to compute  $d_2(x, y)$ , the  $x$ - $y$  second simple shortest path distance (2-SiSP distance, see Section 1.1.2), for all pairs of vertices  $x, y \in V$ . For the output, each distance  $d_2(x, y)$  ( $\forall x \in V$ ) must be known at node  $y$ . Unlike DSO, there are no separate rounds for preprocessing and answering queries.

## 1.2 Results

We summarize our results for DSO and 2-APSiSP in a directed weighted graph  $G = (V, E)$  in the CONGEST model. Recall that  $n = |V|$ , and  $D$  is the undirected diameter of  $G$ .

### 1.2.1 Excluded Shortest Paths.

Our first DSO algorithm and our 2-APSiSP algorithm rely on a subroutine to efficiently compute shortest path distances from multiple sources when certain parts of shortest path trees are excluded, one at a time. Specifically, we present an algorithm for the excluded shortest paths problem defined in Section 1.1.4. We present a CONGEST implementation inspired by a sequential algorithm for a single source in [12], and extend the result to multiple sources by using low congestion SSSP procedures combined with efficient random scheduling. We present our results in Section 2. We will refer to this procedure as an *exclude* computation for brevity.

**Theorem 1.** *Given a set of sources  $X \subseteq V$  and an independent set of paths  $\mathcal{R}_x$  for each source  $x \in X$ , we can compute  $d(x, y, P)$  for each  $x \in X, y \in V, P \in \mathcal{R}_x$  in  $\tilde{O}(n)$  rounds. Additionally, the maximum congestion is  $\tilde{O}(|X|)$ .*

### 1.2.2 Distance Sensitivity Oracles (DSOs).

We construct Distance Sensitivity Oracles for  $G$  that preprocess  $G$  and then answer queries of the form  $d(s, t, e)$  for  $s, t \in V, e \in E$ . We present two algorithms to construct a DSO, one with fast preprocessing, and one with fast query responses. We complement these algorithms with unconditional lower bounds on preprocessing rounds.

**DSO with fast query responses.** Our first DSO optimizes query time, answering any query  $d(s, t, e)$  in  $O(D)$  rounds. More specifically, the DSO uses a constant number of broadcasts to answer a query, after which all nodes know  $d(s, t, e)$ . To answer a batch of  $k$  different queries, we pipeline the broadcasts for each query, answering them in  $O(k + D)$  rounds. This DSO uses  $\tilde{O}(n^{3/2})$  rounds for preprocessing, and is described in Section 3.1. The algorithm performs  $\tilde{O}(\sqrt{n})$  exclude computations from every vertex, such that any query  $d(s, t, e)$  can be answered using  $O(1)$  excluded shortest paths distances stored at different vertices. Thus, once the query is known, we broadcast the relevant distances and compute  $d(s, t, e)$  at the target vertex  $t$  in  $O(D)$  rounds. We use two types of exclude computations: (i) to directly compute  $d(s, t, e)$  when edge  $e$  is within  $\sqrt{n}$  hops of  $s$  or  $t$ , and (ii) to exclude independent segments of each out-shortest path tree  $T_s$ , and combine appropriate excluded shortest path distances to compute  $d(s, t, e)$  if  $e$  is not within  $\sqrt{n}$  hops of  $s$  or  $t$ .

**Theorem 2.** *We can construct a DSO for directed weighted graphs that takes  $\tilde{O}(n^{3/2})$  preprocessing rounds, and then answers a batch of any  $k$  queries of the form  $d(s, t, e)$  in  $O(k + D)$  rounds.*

**DSO with fast preprocessing.** We present another DSO construction that prioritizes preprocessing time, and takes  $\tilde{O}(n)$  rounds. Each query  $d(s, t, e)$  takes  $\tilde{O}(\sqrt{n} + D)$  rounds to answer. To answer a batch of queries, we can pipeline part of the query procedure, and answering  $k$  queries takes  $\tilde{O}(k\sqrt{n} + D)$  rounds. We describe this algorithm in Section 3.2. Note that answering  $k$  queries  $d(s, t, e)$  without any preprocessing can be done with  $k$  SSSP computations ( $\tilde{O}(k \cdot (\sqrt{n} + n^{2/5} D^{2/5} + D))$  rounds [9]), and our algorithm improves on this bound for all  $k \geq 1$ .

In order to reduce preprocessing rounds to  $\tilde{O}(n)$ , we cannot perform too many exclude computations at every node. Instead, we handle short replacement paths of hop-length  $\leq \sqrt{n}$  at query time using a  $\sqrt{n}$ -hop Bellman-Ford procedure. For edges that are further away, we construct randomly sampled graphs where edges in  $G$  are included with certain probability. In these graphs, we perform SSSP computations only from certain sampled vertices, which allows us to compute long replacement path distances at query time.

**Theorem 3.** *We can construct a DSO for directed weighted graphs that takes  $\tilde{O}(n)$  preprocessing rounds, and then answers a batch of any  $k$  queries of the form  $d(s, t, e)$  in  $\tilde{O}(k\sqrt{n} + D)$  rounds.*

Our algorithms focus on computing the distance value of a given query  $d(s, t, e)$ , but we can also readily augment our algorithms so that each vertex on the replacement path from  $x$  to  $y$  avoiding  $e$  knows the next vertex on that path (since each vertex can identify the next vertex in a shortest path after SSSP or APSP computation in  $O(1)$  rounds).

**DSO Lower Bounds.** We show a lower bound on the preprocessing rounds used by a DSO that achieves a given query response round complexity. We consider algorithms that answer a batch of  $k$  queries in  $O(k \cdot Q)$  rounds where  $Q = o(\sqrt{n}/\log n)$ . This covers the salient range of query response complexities, since we can use SSSP to answer queries with no preprocessing and the current best CONGEST lower bound for SSSP is  $\Omega(\sqrt{n} + D)$ . Our lower bound applies even for directed unweighted graphs with diameter  $\log n$ .

**Theorem 4.** *Consider a DSO algorithm for directed unweighted graph  $G = (V, E)$  on  $n$  nodes that performs  $P$  rounds of preprocessing and then answers any batch of  $k \leq \frac{n}{Q^2 \log^2 n}$  queries in  $O(k \cdot Q)$  rounds, where  $Q = o\left(\frac{\sqrt{n}}{\log n}\right)$ . Then,  $P$  is  $\tilde{\Omega}\left(\frac{n}{Q}\right)$ .*

For the setup in Theorem 2, for a DSO algorithm that answers  $k$  queries in  $O(k + D)$  rounds, we show a lower bound of  $\tilde{\Omega}(n)$  preprocessing rounds. Complementing Theorem 3, we show a lower bound of  $\tilde{\Omega}(\sqrt{n})$  preprocessing rounds for a DSO algorithm that answers  $k$  queries in  $o\left(k\frac{\sqrt{n}}{\log n} + D\right)$  rounds. We present these lower bounds for directed unweighted graphs and they trivially apply to directed weighted graphs as well.

Our main technique to obtain this lower bound is to adapt a CONGEST lower bound for SSSP given in [33]. We generalize their construction to  $k$  sources for any  $1 \leq k \leq \frac{n}{2}$ , obtaining a lower bound of  $\Omega\left(\frac{\sqrt{nk}}{\log n}\right)$  for the problem of computing shortest path distances from  $k$  sources ( $k$ -source SSSP) in directed unweighted graphs. We note that CONGEST algorithms for  $k$ -source SSSP have been studied in [24], and our results show that the round complexity  $\tilde{O}(\sqrt{nk} + D)$  in [24] is almost optimal for  $k \geq n^{1/3}$ .

### 1.2.3 All Pairs Second Simple Shortest Paths (2-APSiSP).

We present an algorithm for 2-APSiSP that takes  $\tilde{O}(n)$  rounds to compute the second simple shortest path distance  $d_2(s, t)$  for all pairs of vertices  $s, t \in V$ . Our algorithm builds on a characterization of the second simple shortest path used in a sequential algorithm [1]. We use one exclude computation from each vertex, followed by non-trivial local computation in order to compute all 2-APSiSP distances. We present this algorithm in Section 4.

**Theorem 5.** *We can compute 2-APSiSP for directed weighted graphs in  $\tilde{O}(n)$  rounds.*

**2-APSiSP Lower Bound.** We present a lower bound of  $\tilde{\Omega}(n)$  for computing 2-APSiSP in undirected unweighted graphs (which trivially applies to directed or weighted graphs), proving that our  $\tilde{O}(n)$  round algorithm is optimal up to polylog factors for any class of graphs: directed or undirected, weighted or unweighted. Our lower bound also applies to constant diameter graphs. An  $\tilde{\Omega}(n)$  CONGEST lower bound for computing directed weighted 2-SiSP for one pair was shown in [25] and trivially applies to 2-APSiSP as well. But our lower bound is stronger as the bound in [25] applies only to directed weighted graphs and it requires the input pair of vertices to have a  $\Theta(n)$ -hop shortest path.

**Theorem 6.** *Given an undirected unweighted graph  $G = (V, E)$ , computing 2-APSiSP requires  $\tilde{\Omega}(n)$  rounds. The lower bound holds even when  $G$  has constant diameter.*

We prove this bound by first proving an  $\tilde{\Omega}(n)$  CONGEST lower bound for APSP. Such a lower bound was previously shown in [26], but we give an alternate proof using a reduction from set disjointness. Specifically, we present a graph construction for a reduction from set disjointness to the problem of computing 2-SiSP distances between  $n$  given pairs of vertices, thus giving a 2-APSiSP lower bound.

### 1.3 Prior Work

**Distance Sensitivity Oracle** The DSO problem has been studied extensively in the sequential setting [12, 6, 34, 18, 10, 19]. No non-trivial algorithms for DSO have been studied in the distributed CONGEST setting, but some related problems have received attention. For computing replacement path distances between a single pair of vertices, algorithms and lower bounds were presented in [25] for directed and undirected, weighted and unweighted graphs with nearly optimal bounds in most cases: these include a  $\tilde{\Theta}(n)$  bound (tight up to polylog factors) for directed weighted graphs, and bounds nearly matching *SSSP* round complexity for undirected graphs. A distributed  $O(D \log n)$  algorithm for single source replacement paths in undirected unweighted graphs was given in [15]. Distributed constructions of fault-tolerant preservers and spanners have been studied [8, 27, 13, 28] which construct a sparse subgraph that exactly or approximately preserves replacement path distances. These constructions do not appear to give an efficient procedure to compute replacement path or DSO distances.

**All Pairs Second Simple Shortest Paths** For a single pair of vertices, an  $\tilde{O}(mn)$  sequential time algorithm for the 2-SiSP problem in directed weighted graphs was given in [35]. For 2-APSiSP, a sequential  $\tilde{O}(mn)$  time algorithm matching the 2-SiSP running time was given in [1]. A fine-grained sequential lower bound of  $\Omega(mn)$  complementing these upper bounds was shown in [2] for directed weighted 2-SiSP. In the distributed setting, the problem of computing 2-SiSP for a single pair has been studied in [25] with both upper and lower bounds: these include nearly tight  $\tilde{\Theta}(n)$  bound for directed weighted graphs, and  $\Theta(SSSP)$  bound for undirected graphs.

## 2 Distributed Excluded Shortest Paths

In this section, we present algorithms for the excluded shortest paths problem defined in Section 1.1.4. We are given a graph  $G = (V, E)$ , set of sources  $X \subseteq V$  and an independent set of paths  $\mathcal{R}_x$  for each source  $x$ , and we need to compute  $d(x, y, P)$  for each  $x \in X, y \in V$  and  $P \in \mathcal{R}_x$ . We first consider the single source case, and then extend to multiple sources.

For a single source, we build on a sequential method given in [12], which introduced this problem. From the single source  $x \in V$ , consider a path  $P \in \mathcal{R}_x$ . Recall that  $T_x(P)$  denotes the vertices under the subtree of  $T_x$  rooted at the vertex in  $P$  furthest from  $x$  — the removal of  $P$  only affects distances of vertices in  $T_x(P)$ . Consider a vertex  $z$  in  $T_x(P)$ . The method

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**Algorithm 1** Single Source Excluded Shortest Paths

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**Input:** Graph  $G = (V, E)$ , source  $x \in V$ , independent set of paths  $\mathcal{R}_x$ .

**Output:** Compute distance  $d(x, y, P)$  for each  $y \in V, P \in \mathcal{R}_x$ .

- 1: Compute SSSP from  $x$  to compute  $d(x, y)$  at  $y$  for each  $y \in V$ .
  - 2: Using a downcast from  $x$  along  $T_x$ , communicate identity of  $P \in \mathcal{R}_x$  to each  $z \in T_x(P)$ .
  - 3: Each vertex  $y$  communicates  $d(x, y)$  to its neighbors  $N_{out}(y)$ . If  $y$  is in a subtree  $T_x(P)$  (as identified in line 2), it communicates the identity of  $P$  as well to its neighbors.
  - 4: Each vertex  $z \in T_x(P)$  locally computes  $d^*(x, z) = \min_{v \in N_{in}(z), v \notin T_x(P)} (d(x, z) + w(z, v))$ . The check for  $v \notin T_x(P)$  is done locally using the information from line 3.
  - 5: Perform SSSP from  $x$  on  $G$  with edges in  $\mathcal{R}_x$  removed, and added  $x$ - $z$  edge with weight  $d^*(x, z)$ . These added edges are simulated locally at  $z$  in the course of the SSSP algorithm.
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in [12] adds an edge from  $x$  to  $z$  (we call this the  $x$ - $z$  edge) that captures the minimum distance of a path from  $x$  to  $z$  that does not use any vertex in  $T_x(P)$ . This edge has weight  $d^*(x, z) = \min_{v \in N_{in}(z), v \notin T_x(P)} d(x, v) + w(v, z)$ . We compute the distance  $d(x, z, P)$  by computing SSSP from  $x$  in the graph with these  $x$ - $z$  edges added, and all  $P \in \mathcal{R}_x$  removed.

Algorithm 1 implements this approach in the CONGEST setting. In line 1, we compute SSSP distances from source  $x$ . In line 2, we notify each vertex under the subtree  $T_x(P)$  of an excluded path  $P \in \mathcal{R}$ . By the problem definition, each edge knows the path  $P$  to which it belongs if any, so we only require one downcast along  $T_x$ . The downcast takes  $O(n)$  rounds and  $O(1)$  congestion. Each vertex  $y$  sends distance  $d(x, y)$ , and identity of  $P$  if  $y$  is in  $T_x(P)$  to all its neighbors in line 3. This is an  $O(1)$  round operation since  $\mathcal{R}$  is independent so there is at most one  $T_x(P)$  that  $y$  belongs to. Using this neighbor information, we compute the weight  $d^*(x, z)$  of the added  $x$ - $z$  edge locally at  $z$  in line 4. Finally, the SSSP computation in line 5 computes all excluded shortest paths distances, and distance  $d(x, z, P)$  is known at vertex  $z$ .

In line 1, we use the low congestion SSSP algorithm of [16] for directed weighted graphs which takes  $\tilde{O}(n)$  rounds and has  $\tilde{O}(1)$  congestion. Since all other steps have  $O(n)$  rounds and  $O(1)$  congestion, our total round complexity is  $\tilde{O}(n)$  rounds and  $\tilde{O}(1)$  congestion. Although we focus on algorithms for directed weighted graphs, if the graph is directed and unweighted we can remove the polylog factors to obtain a bound of  $O(n)$  rounds and  $O(1)$  congestion. For this we use  $O(n)$ -round BFS to compute shortest path distances in line 1, and simulate the  $x$ - $z$  weighted edges (which have weight  $d^*(x, z) < n$ ) added in line 5 locally at  $z$  by assuming a message from  $x$  arrives at round  $d^*(x, z)$  of the BFS.

**Multiple-Source Excluded Shortest Paths.** Algorithm 1 gives us an  $\tilde{O}(1)$ -congestion algorithm for computing excluded shortest paths from one source. To compute excluded shortest paths from multiple sources, we can use random scheduling (Section 1.1.3). This leads to Theorem 1 below, which we will use in our DSO and 2-APSiSP algorithms.

**Theorem 1.** (stated in Section 1.2) *Given a set of sources  $X \subseteq V$  and an independent set of paths  $\mathcal{R}_x$  for each source  $x \in X$ , we can compute  $d(x, y, P)$  for each  $x \in X, y \in V, P \in \mathcal{R}_x$  in  $\tilde{O}(n)$  rounds. Additionally, the maximum congestion is  $\tilde{O}(|X|)$ .*

### 3 Distance Sensitivity Oracles

In the DSO problem, we need to preprocess an input graph  $G = (V, E)$  so that we can answer query  $d(x, y, e)$  for any pair of vertices  $x, y \in V$ , edge  $e \in E$  to be avoided. We present algorithms for directed weighted graphs, though they trivially also apply to undirected or unweighted graphs. Note that with no preprocessing, a query can be answered in  $O(SSSP) = \tilde{O}(\sqrt{n} + n^{2/5+o(1)}D^{2/5} + D)$  rounds [9]. We present two algorithms that beat this bound in Sections 3.1 and 3.2. We complement these algorithms with lower bounds in Section 3.3.

In both our algorithms we assume that  $e$  is on a  $x$ - $y$  shortest path. As both algorithms

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**Algorithm 2** Directed Weighted DSO with fast query:  $\tilde{O}(n^{3/2})$  preprocessing and  $\tilde{O}(D)$  query

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**Input:** Graph  $G = (V, E)$ .

**Output:** Answer queries for replacement path distance  $d(x, y, e)$  for each  $x, y \in V, e \in E$ .

- 1:  $\triangleright$  *Preprocessing Algorithm*  $\triangleleft$
  - 2: Compute APSP on  $G$  and the reversed graph  $G^r$ , remembering parent nodes.
  - 3: Perform excluded shortest path computations from each vertex  $x \in V$ , with sets of edges  $R_x^{(1)}, \dots, R_x^{(\sqrt{n})}$  where  $R_x^{(i)}$  is the set of edges  $i$  hops from  $x$  in the (out-)shortest path tree  $T_x$  rooted at  $x$ . Repeat in  $G^r$ .
  - 4: **for**  $x \in V$  **do**
  - 5:  $\triangleright$  *Tree cutting procedure on  $T_x$*   $\triangleleft$
  - 6: Perform an upcast along  $T_x$ , computing size and depth of each vertex, and identify level vertices of both types. Perform an additional upcast and downcast, to identify all *type 1* level vertices. (details described in Section 3.1.1)
  - 7: Perform a downcast along  $T_x$  so that each edge  $e \in T_x$  knows the closest level vertex (if there is one) on the path from  $x$  to  $e$ . Repeat in  $G^r$ . Each edge identifies the subpath between interval vertices it belongs to.
  - 8: Perform exclude computations from  $x$  with the  $O(\sqrt{n})$  sets of independent paths identified in line 7.
  - 9:  $\triangleright$  *Query Algorithm: Given query  $d(x, y, e)$*   $\triangleleft$
  - 10:  $\triangleright$   *$e$  is within  $h$  hops from  $x$  or  $y$  in  $T_x$ : use excluded edge distances from line 3*  $\triangleleft$
  - 11: If  $d(x, y, e)$  has been computed in line 3 at  $x$  or  $y$ , broadcast  $d(x, y, e)$  as the answer.
  - 12:  $\triangleright$   *$e$  is not within  $h$  hops from  $x$  or  $y$  in  $T_x$ : use excluded interval distances from line 8*  $\triangleleft$
  - 13: From line 7,  $e$  locally determines  $a$  as the closest level vertex on the path from  $x$  to  $e$ , and  $b$  as the closest level vertex on the path from  $e$  to  $y$ .  $e$  broadcasts the identities of  $a$  and  $b$ .
  - 14: Then,  $a$  broadcasts  $d(a, y, e)$  and  $x$  broadcasts  $d(x, a), d(x, b, e)$ . Distances  $d(a, y, e)$  and  $d(x, b, e)$  were computed in line 3.
  - 15: Finally,  $y$  broadcasts  $d(x, y, e) = \min(d(x, a) + d(a, y, e), d(x, b, e) + d(b, y), d(x, y, [a, b]))$  as the answer. Distance  $d(x, y, [a, b])$  was computed in line 8.
- 

first compute APSP distances, we can readily check if  $e = (z, z')$  is on a  $x$ - $y$  shortest path by checking if  $d(x, y) = d(x, z) + w(e) + d(z', y)$ , and return  $d(x, y, e) = d(x, y)$  if not.

### 3.1 DSO with fast query responses

Our first method for directed weighted DSO with fast query responses is described in Algorithm 2, which takes  $\tilde{O}(n^{3/2})$  rounds for preprocessing and then answers a batch of any  $k$  queries  $d(x, y, e)$  in  $O(k + D)$  rounds.

Our algorithm builds on a sequential DSO algorithm in [12] that runs in time  $O(mn^{1.5} + n^{2.5} \log n)$  and answers queries in  $O(1)$  time. The main idea is to consider replacement distances  $d(x, y, e)$  of two forms in the preprocessing phase. If  $e$  is within  $\sqrt{n}$  hops of  $x$  in its shortest path tree  $T_x$ , we perform exclude computations from each node, excluding edges within  $\sqrt{n}$  depth in its outgoing shortest path tree. We do the same on the reversed graph. For the remaining distances, when  $e$  is at least  $\sqrt{n}$  hops away from both  $x$  and  $y$ , we make use of interval computations. We use a *tree cutting procedure* described below on each shortest path tree  $T_x$ , which designates certain vertices as *level vertices* such that the subpaths of  $T_x$  in the intervals between level vertices can be arranged into  $O(\sqrt{n})$  sets of independent paths. We ensure that any edge  $e$  that is  $\sqrt{n}$  hops away from both  $x$  and  $y$  is flanked by some pair of level vertices w.h.p. in  $n$ , and use the distances when these intervals are excluded to answer query  $d(x, y, e)$  as shown below.



### 3.1.1 Tree cutting procedure

In order to identify appropriate level vertices, we use the following method from [12]: In shortest path tree  $T_x$ , let the size of vertex  $y$  be the number of nodes in the subtree of  $T_x$  rooted at  $y$ . If a vertex  $y$  has more than one child with size  $\geq \sqrt{n}$ , all vertices at the same depth as  $y$  in  $T_x$  are designated *type 1* level vertices. As noted in [12] there are at most  $\sqrt{n}$  such  $y$ . We are only concerned with  $\sqrt{n}$ -length paths, so we do not need to consider nodes of size  $< \sqrt{n}$  when computing *type 1* level vertices. Additionally, all vertices at depth  $i\sqrt{n}$  in  $T_x$ , for  $i = 1, 2, \dots, \sqrt{n}$  are designated *type 2* level vertices so that any path of length  $\geq \sqrt{n}$  contains a level vertex. Specifically, if edge  $e$  on a  $x$ - $y$  shortest path is  $\sqrt{n}$  hops away from both  $x$  and  $y$ ,  $e$  is flanked by some pair of level vertices on the  $x$ - $y$  path. Combining both types, level vertices occur at  $\leq 2\sqrt{n}$  depth values, and vertices at these depths partition the tree into  $O(\sqrt{n})$  intervals. The set of paths in a given interval with level vertex endpoints on either side are independent (as shown in [12]). So, we have arranged the subpaths of the tree between level vertices into  $O(\sqrt{n})$  sets of independent paths.

In lines 6-8 of Algorithm 2, we use a distributed implementation of this method as follows: In line 6, we perform an upcast along  $T_x$  in  $O(n)$  rounds, so that each vertex knows its size and depth in  $T_x$ , and use this to identify level vertices of both types described in the paragraph above. With an additional upcast and downcast, all vertices at the same depth as some type 1 level vertex also identifies itself as a level vertex. After the downcast in line 7, each edge knows its closest level vertex in either direction. So, in the exclude computation in line 8, each edge knows which interval subpath it is a part of. We use  $O(n)$  rounds and  $O(1)$  congestion per tree  $T_x$  for the computation in lines 6,7.

### 3.1.2 Analysis of Algorithm 2

We utilize the multiple excluded shortest paths algorithm from Section 2 for efficiently excluding paths in shortest path trees. We perform a total of  $\sqrt{n}$  exclude computations per source  $x$  in lines 3,8, which takes  $\tilde{O}(n^{3/2})$  rounds using Theorem 1. In the query algorithm, given query  $d(x, y, e)$ , we identify the level vertices flanking  $e$  if they exist in line 13. We broadcast the excluded shortest paths distances necessary to compute  $d(x, y, e)$  in lines 11,14 in  $O(D)$  rounds, and compute the result at  $y$ .

**Theorem 2.** (stated in Section 1.2) *Algorithm 2 takes  $\tilde{O}(n^{3/2})$  preprocessing rounds, and then answers a batch of any  $k$  queries of the form  $d(s, t, e)$  in  $O(k + D)$  rounds.*

*Proof. Correctness:* Let  $P_{xy}$  be a replacement path from  $x$  to  $y$  when  $e$  is removed. If  $e$  is within  $h = \sqrt{n}$  hops from  $x$  or  $y$  in the shortest path tree, then  $d(x, y, e)$  is computed at  $y$  in line 3 and the query is answered in line 11.

Now, consider the other case where  $e$  is not within  $h$  hops of  $x$  or  $y$  on the  $x$ - $y$  shortest path. After the cutting procedure on  $T_x$ , there are level vertices  $a, b$  such that  $a$  is on the shortest path from  $x$  to  $e$  and  $b$  is on the shortest path from  $e$  to  $y$ . Additionally, the interval subpath from  $a$  to  $b$  has at most  $\sqrt{n}$  hops, so  $e$  is within  $\sqrt{n}$  hops of both  $a$  and  $b$  in their respective shortest path trees. Thus,  $d(a, y, e)$  is computed at  $y$  in line 3 and similarly  $d(x, b, e)$  is computed at  $x$  in the reversed graph. After the identities of  $a, b$  are broadcast, the replacement path distance is correctly computed in line 15 — the first term is correct when  $P_{xy}$  passes through  $a$  and the second is correct when it passes through  $b$ . If  $P_{xy}$  passes through neither  $a$  or  $b$ , it skips the whole interval  $[a, b]$ , and the distance is correctly computed as  $d(x, y, [a, b])$ .

**Round Complexity:** Computing APSP takes  $\tilde{O}(n)$  rounds. We perform  $\tilde{O}(n^{3/2})$  different exclude computations in lines 3 and 8. Using Theorem 1 this takes  $\tilde{O}(n^{3/2})$  rounds. The downcast in line 7 takes  $O(n)$  rounds and  $O(1)$  congestion per vertex, and can be scheduled for all  $x$  in  $\tilde{O}(n)$  rounds. The query algorithm broadcasts  $O(1)$  values and then broadcasts the correct answer, so it takes  $O(D)$  rounds with  $O(1)$  congestion per edge. Thus, we can answer a batch of  $k$  queries by pipelining all broadcasts in  $O(k + D)$  rounds.  $\square$

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**Algorithm 3** Directed Weighted DSO with fast preprocessing:  $\tilde{O}(n)$  preprocessing and  $\tilde{O}(\sqrt{n} + D)$  query

---

**Input:** Graph  $G = (V, E)$ .

**Output:** Answer query for replacement path distance  $d(x, y, e)$  for each  $x, y \in V, e \in E$ .

```

1:  $\triangleright$  Preprocessing Algorithm ◁
2: for  $j = \log(\sqrt{n}), \dots, \log n$  do
3:    $\triangleright$  Preprocessing for replacement paths of hop-length  $[2^j, 2^{j+1}]$ , where  $2^j \geq \sqrt{n}$  ◁
4:   Each vertex  $s \in V$  is sampled and added to  $S_j$  with probability  $\frac{c \log n}{2^j}$ .
5:   Let  $h = 2^{j+1}$ . Sample  $\tilde{O}(h)$  graphs  $G_i^j$  as in Lemma 7 by randomly sampling each edge
   with probability  $1 - \frac{1}{h}$ .  $\triangleright$  Each edge locally samples itself
6:   For each  $e \in G$ ,  $I_e^j \leftarrow \{(j, i) \mid e \notin G_i^j\}$   $\triangleright$  Computed locally at endpoints of  $e$ 
7:   Compute SSSP from each vertex in  $S_j$ , on each  $G_i^j$  (and reversed graph). The distance
   in graph  $G_i^j$  is denoted  $d_i^j$ . Distances  $d_i^j(s, x), d_i^j(x, s)$  are computed at  $x$  for each  $s \in$ 
    $S_j, x \in V$ .
8:  $\triangleright$  Query Algorithm: Given query  $d(x, y, e)$  ◁
9:  $\triangleright$  Find best replacement path of hop-length  $\leq \sqrt{n}$  ◁
10: Perform a distributed Bellman-Ford computation in  $G - \{e\}$  starting from  $x$ , up to  $\sqrt{n}$  hops.
   This computes  $d_{\sqrt{n}}(x, y, e)$  at  $y$ .
11:  $\triangleright$  Find best replacement path of hop-length  $> \sqrt{n}$  ◁
12: for  $j = \log(\sqrt{n}), \dots, \log n$  do
13:    $\triangleright$  Handle paths of hop-length  $[2^j, 2^{j+1}]$  ◁
14:   Endpoint of  $e$  broadcasts  $I_e^j$ .
15:    $x$  broadcasts  $\{d_i^j(x, s) \mid (j, i) \in I_e^j, s \in S_j\}$ , i.e. its distance to sampled vertices in  $S_j$  in
   graphs where  $e$  is not present.
16:   for  $s \in S_j$  do  $\triangleright$  Local computation at  $y$ 
17:      $\triangleright$   $d_s$  is the best  $x$ - $y$  replacement path distance through  $s$  ◁
18:     Compute  $d_s = \min_{(j, i) \in I_e^j} d_i^j(x, s) + d_i^j(s, y)$ 
19: Compute  $d(x, y, e) = \min(d_{\sqrt{n}}(x, y, e), \min_{s \in V} d_s)$   $\triangleright$  Local computation at  $y$ 

```

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## 3.2 DSO with fast preprocessing

In this section, we present Algorithm 3, which constructs a DSO on directed weighted graphs with  $\tilde{O}(n)$  rounds for preprocessing. After preprocessing, the DSO answers a batch of  $k$  queries in  $\tilde{O}(k \cdot \sqrt{n} + D)$  rounds. The algorithm uses a graph sampling approach to compute hop-limited replacement paths, similar to ideas used in sequential DSO constructions [34, 32]. We first describe this method (Section 3.2.1), and then describe our distributed algorithm which uses this graph sampling method along with vertex sampling to achieve efficient preprocessing.

### 3.2.1 Hop-limited replacement paths

Given a hop parameter  $h$ , and a graph  $G = (V, E)$  along with a set of sources  $S \subseteq V$ , we wish to compute  $h$ -hop replacement path distances  $d_h(s, x, e)$  for  $s \in S, x \in V, e \in E$ . Here,  $d_h(s, x, e)$  denotes the minimum weight of a path containing up to  $h$  edges from  $s$  to  $x$  that does not contain edge  $e$ . The following lemma can be established by adapting proofs in [34].

**Lemma 7.** *Sample  $\tilde{h} = 12h \log n$  graphs  $G_1, \dots, G_{\tilde{h}}$  as subgraphs of  $G$  by including each of  $G$  with probability  $1 - \frac{1}{h}$ . Then,*

- A. *W.h.p. in  $n$ , there are  $O(\log^2 n)$  graphs  $G_i$  such that  $e \notin G_i$ .*
- B. *Let  $P_{sx}^e$  be a  $h$ -hop replacement path from  $s$  to  $x$  avoiding  $e$ . W.h.p. in  $n$ , there is at least one graph  $G_i$  such that  $G_i$  contains all edges in  $P_{sx}^e$  but does not contain  $e$ .*

### 3.2.2 CONGEST DSO with fast preprocessing

We now present Algorithm 3 for DSO that takes  $\tilde{O}(n)$  preprocessing time and  $\tilde{O}(\sqrt{n} + D)$  query time. To obtain fast preprocessing time, we only deal with replacement paths of hop-length  $\geq \sqrt{n}$  during preprocessing. Shorter replacement paths are computed at query time using  $\sqrt{n}$  steps of Bellman-Ford from the start vertex with the appropriate edge deleted.

To compute replacement paths of hop-length  $[2^j, 2^{j+1}]$ , for  $j \geq \frac{1}{2} \log n$ , we use the hop-limited replacement path result in Lemma 7 with  $h = 2^{j+1}$ . In line 5, we sample  $\tilde{O}(2^{j+1})$  graphs  $G_i^j$  by including edges with probability  $1 - \frac{1}{2^{j+1}}$ . To compute replacement path distance  $d(x, y, e)$  for a particular edge  $e$ , we look at the sampled graphs  $G_i^j$  where  $e$  is not present. By Lemma 7.A, computing the minimum over  $x$ - $y$  shortest path distances up to hop-length  $2^{j+1}$  in these graphs will give the correct distance  $d_{2^{j+1}}(x, y, e)$  w.h.p. in  $n$ . We collect the identities of these graphs where  $e$  is not present in a set  $I_e^j$  in line 6, and store it locally at an endpoint of  $e$ . Instead of simply computing APSP distances in each sampled graph  $G_i^j$  (which would take  $\tilde{\Theta}(n \cdot 2^j)$  rounds) we instead sample vertices in  $G_i^j$  and compute hop-limited replacement paths through sampled vertices. Using a sampling probability of  $\tilde{\Theta}(\frac{1}{2^j})$  as in line 4, any path of hop-length at least  $2^j$  will contain a sampled vertex w.h.p. in  $n$ , and we compute  $d(x, y, e)$  through this sampled vertex. Specifically, we compute  $2^{j+1}$ -hop distances to and from sampled vertices  $s$  in graphs  $G_i^j$  in line 7. In the appropriate  $G_i^j$  which contains the entire replacement path and does not contain  $e$ , the distances  $d_{2^{j+1}}(x, s, e)$  and  $d_{2^{j+1}}(s, y, e)$  are correctly computed.

To answer a query  $d(x, y, e)$ , we use Bellman-Ford if the replacement path has  $\leq \sqrt{n}$  hops in line 10. To compute replacement path if it has  $> \sqrt{n}$  hops,  $e$  broadcasts  $I_e^j$  in line 14, and the minimum  $x$ - $y$  replacement path distance through each sampled vertex in  $G_i^j$  with  $e$  deleted is computed in line 18. The minimum will be  $d(x, y, e) = d(x, s, e) + d(s, y, e)$  for the sampled vertex  $s$  on the  $x$ - $y$  replacement path in the graph  $G_i^j$  containing this whole path — both of which exist w.h.p. in  $n$ . Note that this step involves  $x$  broadcasting one distance value for each  $s$ . Even for replacement paths of hop-length  $\leq \sqrt{n}$ , (i.e.,  $j \leq \frac{1}{2} \log n$ ) we could compute the shortest paths through sampled vertices efficiently within the  $\tilde{O}(n)$  bound. However the number of sampled vertices would exceed  $\sqrt{n}$ , and broadcasting the distances to each sampled vertex in order to answer a query would take  $\omega(\sqrt{n})$  rounds.

**Theorem 3.** (stated in Section 1.2) *Algorithm 3 takes  $\tilde{O}(n)$  preprocessing rounds, and then answers a batch of any  $k$  queries of the form  $d(s, t, e)$  in  $\tilde{O}(k\sqrt{n} + D)$  rounds.*

*Proof. Correctness:* If the  $x$ - $y$  replacement path has hop-length  $\leq \sqrt{n}$ , the distance is correctly computed in line 10 using the Bellman-Ford computation from  $x$  up to  $\sqrt{n}$  hops. The algorithm deals with replacement paths of hop-length  $[2^j, 2^{j+1}]$  separately, for  $2^j \geq \sqrt{n}$ . Fix one such replacement path  $\mathcal{P}$  from  $x$  to  $y$  avoiding edge  $e$ , with hop length  $2^j \leq h(\mathcal{P}) < 2^{j+1}$  and weight  $w(\mathcal{P})$ . In the preprocessing algorithm, consider the  $j$ 'th iteration of the loop (lines 2-7). Since we sample vertices into  $S_j$  with probability  $\tilde{O}(\frac{1}{2^j})$ , w.h.p in  $n$  there is at least one vertex  $s$  from  $S_j$  that is on  $\mathcal{P}$ . According to Lemma 7.A, w.h.p. in  $n$ , at least one of the sampled subgraphs  $G_i^j$  contains all edges of  $\mathcal{P}$  and does not contain  $e$ . After line 7, where shortest path distances to and from  $s \in S_j$  are computed in the sampled graphs, the distances  $d_i^j(s, y)$  and  $d_i^j(x, s)$  are correctly computed. In this particular  $G_i^j$ , we have  $w(\mathcal{P}) = d_i^j(x, s) + d_i^j(s, y)$ .

In order to answer query  $d(x, y, e)$ , we first identify if edge  $e$  is present in graph  $G_i^j$ , using the set  $I_e^j$  computed in line 7 and broadcast in line 14. In line 18, we only consider shortest path distances in graphs not containing  $e$ , so all distances  $d_s$  correspond to valid  $x$ - $y$  paths not containing  $e$ . For the particular  $G_i^j$  that contains the path  $\mathcal{P}$ , we correctly compute the distance  $d(x, y, e) = d_i^j(x, s) + d_i^j(s, y)$ . Any  $G_i^j$  not containing all edges of  $\mathcal{P}$  would only have higher  $s$ - $t$  distance, so the minimum computation in 18 gives the correct result (w.h.p. in  $n$ ).

**Round Complexity:** We first analyze the preprocessing algorithm. Consider an iteration  $j$ : lines 4-6 only involve local computation, and line 7 performs SSSP from  $|S_j|$  vertices on  $\tilde{O}(h)$

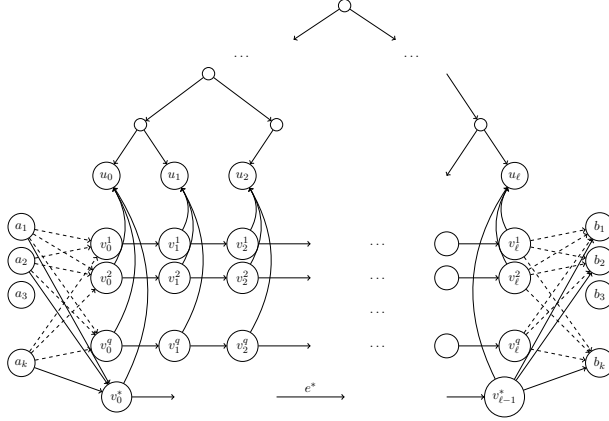


Figure 1: Lower Bound for DSO Preprocessing Time

graphs. Using the low congestion SSSP algorithm from [16], the total congestion over all SSSP computation is at most  $\tilde{O}(|S_k|h) = \tilde{O}(n)$  and using random scheduling, line 7 takes  $\tilde{O}(n)$  rounds. Repeating this for  $\log n$  iterations for each  $j = 1$  to  $\log n$  takes  $\tilde{O}(n)$  rounds.

Now, to answer a query  $d(x, y, e)$ , computing Bellman-Ford up to  $\sqrt{n}$  hops in line 10 takes  $O(\sqrt{n})$  rounds. The set  $I_e^j$  is broadcast in line 14. This set  $I_e^j$  has size  $O(\log^2 n)$  (Lemma 7.B), so the broadcast takes  $\tilde{O}(D)$  rounds. Vertex  $x$  then broadcasts a set of size  $|I_e^j| \cdot |S_j| = \tilde{O}(\frac{n}{2^j}) = \tilde{O}(\sqrt{n})$  (since  $2^j \geq \sqrt{n}$ ) in line 15, which takes  $\tilde{O}(\sqrt{n} + D)$  rounds. Lines 18,19 only involve local computation at  $y$ , so the total rounds to answer the query is  $\tilde{O}(\sqrt{n} + D)$ .

To answer a batch of  $k$  queries, we can broadcast the  $\tilde{O}(k\sqrt{n})$  required values in lines 14,15 in  $\tilde{O}(k\sqrt{n} + D)$  rounds, and perform  $k$  Bellman-Ford computations sequentially in  $k\sqrt{n}$  rounds. So, we take a total of  $\tilde{O}(k\sqrt{n} + D)$  rounds.  $\square$

### 3.3 DSO Lower bounds

In this section, we show lower bounds on the preprocessing rounds used by any DSO algorithm with a given query response round complexity. Our proof uses a graph construction to obtain a reduction from set disjointness to the problem of computing  $k$  DSO queries in a directed unweighted graph, building on a  $\tilde{\Omega}(\sqrt{n})$  CONGEST lower bound for SSSP in [33]. This construction also leads to Lemma 10, a lower bound of  $\tilde{\Omega}(\sqrt{nk} + D)$  for computing  $k$ -source shortest paths in a directed unweighted graph, which is tight for  $k \geq n^{1/3}$  (by an upper bound in [24]). These unweighted lower bounds also trivially apply to directed weighted graphs. We start with the following preliminary lemma.

**Lemma 8.** *Consider an algorithm  $\mathcal{A}$  for a directed unweighted graph  $G = (V, E)$  on  $n$  nodes that computes the answer to any batch of  $k \leq n/2$  queries  $d(a_i, b_i, e_i)$  for  $1 \leq i \leq k$ . The answer for query  $i$  must be known to  $b_i$  after the algorithm terminates. Then,  $\mathcal{A}$  must have round complexity  $\Omega\left(\frac{\sqrt{nk}}{\log n}\right)$ , even on graphs of undirected diameter  $\Theta(\log n)$ .*

*Proof. Graph Construction:* We construct directed unweighted graph  $G = (V, E)$ , pictured in Figure 1, as a balanced binary tree with  $\ell$  leaves, numbered  $u_1, \dots, u_\ell$ , along with a set of  $q$  paths  $v_0^i - v_1^i - v_2^i - \dots - v_{\ell-1}^i$  of length  $\ell$ , and a path  $v_0^* - v_1^* - v_2^* - \dots - v_{\ell-1}^*$  of length  $\ell - 1$ . The paths are connected to the leaves of the tree as in the figure, with all edges oriented from path to leaf. Additionally, we have  $2k$  vertices  $a_i, b_i$  for  $1 \leq i \leq k$ . We perform a reduction from  $(kq)$ -bit Set Disjointness, and the edge weights in our construction depends on the inputs  $S_a, S_b$  of the set disjointness instance. We index the input such that  $S_a[i, j]$  represents the  $((i-1) \cdot q + j)$ 'th bit for  $1 \leq i \leq k, 1 \leq j \leq q$  (similarly for  $S_b$ ). The edge  $(a_i, v_0^i)$  is present only if  $S_a[i, j] = 1$ , similarly edge  $(v_{\ell-1}^i, b_i)$  is present only if  $S_b[i, j] = 1$ . All other solid edges in Figure 1 are always present.

Note that  $d(a_i, b_i) = \ell + 1$  due to the  $v_0^* - v_{\ell-1}^*$  path. When an edge  $e^*$  on the  $v_0^* - v_{\ell-1}^*$  path is removed, a replacement path from  $a_i$  to  $b_i$  with weight  $\ell + 2$  exists through  $v_0^i - v_{\ell-1}^i$  if

$S_a[i, j] = S_b[i, j] = 1$  for some  $j$ . Otherwise, either the  $(a_i, v_0^j)$  or  $(v_\ell^j, b_i)$  edge is missing for every  $j$ , and no  $a_i$ - $b_i$  replacement path exists. Thus, we get the following claim.

**Claim 1:** Let  $e^*$  be an edge on the  $v_0^* - v_{\ell-1}^*$  path.  $d(a_i, b_i, e^*) = \ell + 2$  if  $S_a[i][j] = S_b[i][j] = 1$  and  $d(a_i, b_i, e^*) = \infty$  otherwise.

**Claim 2:** If there is an algorithm  $\mathcal{A}$  that computes answer to  $k$  queries  $d(a_i, b_i, e_i)$  in  $R \leq \ell$  rounds, then we can solve  $(kq)$ -bit set disjointness using  $Rp \cdot \Theta(\log n)$  bits, where  $p$  is the height of tree.

The  $\Theta(\log n)$  term is due to the CONGEST bandwidth. The proof of Claim 2 follows from analysis similar to ([33], Theorem 3.1), and is as follows: Consider an  $i$ -left set  $L_i$  which consists of the first  $(\ell - i)$  leaf vertices, path vertices  $v_j^r$  for  $1 \leq r \leq q$  and  $j \leq \ell - i$ , and all tree vertices that are the ancestor of at least one included leaf vertex. Similarly, an  $i$ -right set consists of the last  $(\ell - i)$  leaf vertices, their ancestors in the tree, and path vertices  $v_j^r$  for  $1 \leq r \leq q, j \geq i$ . The idea is that for  $i < \frac{\ell}{2}$ , at round  $i$ , only limited amount of information from  $a_1, a_2, \dots, a_k$  vertices reaches the vertices in  $R_i$ , and similarly only limited information from  $b_1, \dots, b_k$  vertices reaches vertices in  $L_i$ . More specifically, given the states (i.e. received messages and local computation) of vertices in  $R_{i-1}$  at round  $(i - 1)$ , we can compute the states of vertices in  $R_i$  at round  $i$  with an additional  $\leq p$  messages sent from vertices outside  $R_{i-1}$  to vertices in  $R_i$ . A similar claim holds for  $L_i$ , but we will prove this only for  $R_i$ .

Consider a neighbor  $v$  of vertex  $u$  in  $R_i$ . If  $v$  is in  $R_{i-1}$ , then we know the state of  $v$  at round  $(i - 1)$ , and hence the message that it would have sent to  $u$ . Thus, we are only concerned with messages from neighbors  $v$  that are not in  $R_{i-1}$ . As shown in [33], this only happens with tree ancestor vertices where a vertex is in  $R_i$  but one of its children is not in  $R_{i-1}$ . There can be at most one just neighbor at each level of the tree (at the leftmost boundary of  $R_i$ ), so there are at most  $p$  of them. So, given the  $\leq p$  messages sent along these edges, the state of all vertices in  $R_i$  at round  $i$  can be determined.

Now, we consider a simulation of the CONGEST algorithm by Alice and Bob. Initially, Alice has the bits  $S_a$ , and thus knows the edges incident to  $a_j$  (for  $1 \leq j \leq k$ ) and  $v_0^r$  (for  $1 \leq r \leq q$ ). Similarly, Bob has  $S_b$  and knows the edges incident to  $b_j$  and  $v_\ell^r$ . Initially, Alice knows the state of  $L_1$  (which includes  $v_0^r$ 's but not  $v_\ell^r$ 's) and Bob knows the state of  $R_1$ . Observe that if  $i < \frac{\ell}{2}$ ,  $V \setminus R_{t-1} \subseteq L_{t-1}$ . For each round of the CONGEST algorithm, Alice generates the  $\log n$  messages from  $V \setminus R_{t-1}$  that Bob requires to compute  $R_t$  from the state of  $R_{t-1}$ . Similarly, Bob generates the  $p$  messages from  $V \setminus L_{t-1} \subseteq R_{t-1}$  that Alice requires to compute  $L_t$  from  $L_{t-1}$ . Thus, to simulate one round, Alice and Bob exchange  $p \cdot \Theta(\log n)$  bits. To simulate  $R$  rounds of the CONGEST algorithm, Alice and Bob communicate  $\Theta(Rp \log n)$  bits.

To choose our parameters, we use  $p = O(\log n), q = \frac{n-k}{\ell} \geq \frac{n}{2\ell}$  (since  $k < n/2$ ),  $\ell$  will be chosen later. The number of vertices in the graph is  $\Theta(\ell \cdot q + k) = \Theta(n)$ . The undirected diameter of the graph is  $\leq 2p + 4 = \Theta(\log n)$  (using paths through the tree). The number of bits in the set disjointness instance is  $kq \geq \frac{kn}{2\ell}$ . Using the randomized communication lower bound for set disjointness [31, 5, 21] along with Claim 2, we obtain: If  $R(n) \leq \ell$ , then  $R(n) \cdot \log^2 n = \Omega(kq) \Rightarrow R(n) = \Omega\left(\frac{kn}{\ell \log^2 n}\right)$ . Combining the two bounds,  $R(n) = \Omega\left(\min\left(\ell, \frac{kn}{\ell \log^2 n}\right)\right)$ . We balance these terms by choosing  $\ell = \frac{\sqrt{nk}}{\log n}$ , which gives  $R(n) = \Omega\left(\frac{\sqrt{nk}}{\log n}\right)$ .  $\square$

We now apply this lemma to obtain a lower bound for DSO, proving Theorem 4.

**Theorem 4.** (stated in Section 1.2) Consider a DSO algorithm for directed unweighted graph  $G = (V, E)$  on  $n$  nodes that performs  $P$  rounds of preprocessing and then answers any batch of  $k \leq \frac{n}{Q^2 \log^2 n}$  queries in  $O(k \cdot Q)$  rounds, where  $Q = o\left(\frac{\sqrt{n}}{\log n}\right)$ . Then,  $P$  is  $\tilde{\Omega}\left(\frac{n}{Q}\right)$ . This lower bound applies even for graphs of diameter  $\Theta(\log n)$ .

*Proof.* Applying Lemma 8,  $P + k \cdot Q = \Omega\left(\frac{\sqrt{nk}}{\log n}\right)$ . Choosing  $k$  such that  $k \cdot Q = o\left(\frac{\sqrt{nk}}{\log n}\right)$ , would give a lower bound on  $P$ . Thus,  $\sqrt{k} = o\left(\frac{\sqrt{n}}{Q \log n}\right) \Rightarrow k = o\left(\frac{n}{Q^2 \log^2 n}\right)$ . Then,  $k \cdot Q = \frac{n}{Q \log^2 n} =$

$o\left(\frac{\sqrt{nk}}{\log n}\right)$ . Choosing  $k = \Theta\left(\frac{n}{Q^2 \log^3 n}\right)$  for instance, we get  $P = \Omega\left(\frac{\sqrt{nk}}{\log n}\right) = \tilde{\Omega}\left(\frac{n}{Q}\right)$ .  $\square$

We now apply Theorem 4 to obtain lower bounds mirroring the upper bounds in Section 3, with query setups similar to Algorithm 2 (Corollary 9.A) and Algorithm 3 (Corollary 9.B). Algorithm 3 has query response rounds  $\tilde{O}(k\sqrt{n} + D)$ , and the query setup in Corollary 9.B differs from it only by a polylog factor.

**Corollary 9.** *Consider a DSO algorithm  $\mathcal{A}$  for directed unweighted graph  $G = (V, E)$  on  $n$  nodes that uses  $P$  rounds for preprocessing.*

- A. *If  $\mathcal{A}$  can answer  $k$  queries in  $O(k + D)$  rounds, i.e.,  $Q = O(1)$ , then  $P$  is  $\tilde{\Omega}(n)$ .*
- B. *If  $\mathcal{A}$  can answer  $k$  queries in  $o\left(k \cdot \frac{\sqrt{n}}{\log n}\right)$  rounds, i.e.,  $Q = o\left(\frac{\sqrt{n}}{\log n}\right)$ , then  $P$  is  $\tilde{\Omega}(\sqrt{n})$ .*

### 3.3.1 Lower Bound for $k$ -source shortest path

We derive a lower bound for  $k$ -source shortest path by modifying the lower bound construction in the previous section. For this, we remove path  $v_0^* - v_{\ell-1}^*$ . Then, we can readily modify the proof of Lemma 8 for computing the  $k$  distances  $d(a_i, b_i)$  in the modified graph, to obtain the following result.

**Lemma 10.** *Any CONGEST algorithm for directed unweighted graphs that computes shortest path distances from  $k$  sources must take  $\tilde{\Omega}(\sqrt{nk} + D)$  rounds.*

This lower bound is almost tight for  $k \geq n^{1/3}$ , as the  $k$ -source SSSP algorithm of [24] takes  $\tilde{O}(\sqrt{nk} + D)$  rounds for  $k$ -SSSP in directed unweighted graphs. For  $k < n^{1/3}$  the bound is not tight, which is not unexpected as we have a gap between the current best upper and lower bounds even for one source ( $k = 1$ ), i.e.,  $\tilde{O}(\sqrt{n} + n^{2/5+o(1)}D^{2/5} + D)$  [9] and  $\tilde{\Omega}(\sqrt{n} + D)$  [33].

## 4 All Pairs Second Simple Shortest Path

In the 2-APSiSP problem, we are given a graph  $G = (V, E)$  and need to compute the second simple shortest path (2-SiSP) distance  $d_2(x, y)$  for any pair of vertices  $x, y \in V$ . We present a  $\tilde{O}(n)$  round algorithm for 2-APSiSP in directed weighted graphs and show it is almost optimal with a  $\tilde{\Omega}(n)$  round lower bound even for undirected unweighted graphs with constant diameter.

### 4.1 2-APSiSP Algorithm

We present Algorithm 4 for computing 2-APSiSP in directed weighted graphs in  $\tilde{O}(n)$  rounds, such that distance  $d_2(x, y)$  is known at  $y$ . Our algorithm uses the following characterization of 2-SiSP proved in [1].

**Lemma 11.** *Given  $x, y \in V$ , let  $a$  be the vertex after  $x$  on the  $x$ - $y$  shortest path. Let  $d(x, y, (x, a))$  denote the replacement path distance from  $x$  to  $y$  when edge  $(x, a)$  is removed. Then, if  $a \neq y$ , 2-SiSP distance  $d_2(x, y) = \min(d(x, y, (x, a)), w(x, a) + d_2(a, y))$*

Our algorithm performs an exclude computation in line 3 from each  $x \in V$  with all outgoing edges of  $x$  in  $T_x$  as the set of excluded edges. This computes  $d(x, y, (x, a))$  for all  $y \in V$  at  $y$ , where  $(x, a)$  is the first edge on the  $x$ - $y$  shortest path. These  $n$  exclude computations can be performed in  $\tilde{O}(n)$  time (Theorem 1). Then, we order the computations so that  $d_2(z, y)$  is first computed at  $y$  for  $z \in V$  such that  $z$ - $y$  shortest path has hop-length 1, then for  $z$  such that  $z$ - $y$  shortest path has hop-length 2, and so on until all  $d_2(z, y)$  are computed. All this is done locally at  $y$  in lines 4-7, so the total round complexity remains  $\tilde{O}(n)$ .

**Lemma 12.** *Algorithm 4 correctly computes 2-APSiSP distances  $d_2(x, y)$  in  $\tilde{O}(n)$  rounds.*

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**Algorithm 4** Directed Weighted 2-APSiSP

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**Input:** Graph  $G = (V, E)$ .

**Output:** Compute 2-SiSP distance  $d_2(x, y)$  at  $y$  for each  $x, y \in V$ .

- 1: Compute APSP in  $G$  and  $G^r$ , remembering parent nodes.
  - 2: For each vertex  $x$ , find all neighbors  $a$  such that edge  $(x, a)$  is a shortest path. Perform a downcast on  $T_x$ , the out-shortest path tree rooted at  $x$ , so that the identity of  $(x, a)$  and  $w(x, a)$  is known to vertices in subtree of  $T_x$  rooted at  $a$ .
  - 3: Compute excluded shortest paths from  $x$  with the set of edges  $(x, a)$  excluded. Thus, we compute  $d(x, y, (x, a))$  at all  $y$ , where  $(x, a)$  is the first edge of  $x$ - $y$  shortest path.
  - 4: **for**  $y \in V$  **do**  $\triangleright$  Local computation at  $y$
  - 5:      $y$  computes  $d_2(z, y) = d(z, y, (z, y))$  for in-edges  $(z, y)$  such that  $(z, y)$  is a shortest path.
  - 6:     **for up to**  $n$  iterations **do**
  - 7:         For  $z' \in V$  such that  $y$  received distance  $d(z', y, (z', z))$  (in line 3) and  $d_2(z, y)$  has already been computed, compute  $d_2(z', y) = \min(d_2(z, y) + w(z', z), d(z', y, (z', z)))$ .  
       Note that  $w(z', z)$  was sent to  $y$  in line 2 as edge  $(z', z)$  is a shortest path.
- 

*Proof. Correctness:* Given the characterization in Lemma 11, we argue that the computation in lines 4-7 correctly computes all distances  $d_2(y, z)$  at  $y$ . In the sequential algorithm in [1], a global priority queue was used, but here we locally compute all relevant values at each  $y$ . In line 5, we correctly compute  $d_2(z, y)$  where edge  $(z, y)$  is a one-hop shortest path. In the next step in the for loop in line 6, we compute  $d_2(z', y)$  where  $z'$ - $y$  shortest path has two hops. This is because of Lemma 11, and  $(z', z)$  is a shortest path for some  $z$  so  $y$  gets the distance  $d(z', y, (z', z))$  during the computation in line 2. Similarly, we can argue that after  $d_2(z, y)$  has been computed for  $k$ -hop shortest paths  $z$ - $y$ , we compute  $d_2(z', y)$  for  $(k+1)$ -hop shortest paths  $z'$ - $y$  in the next iteration of line 6. After up to  $n$  steps, all  $z$  (with finite  $d_2(z, y)$ ) have their distance  $d_2(z, y)$  computed.

**Round Complexity:** APSP takes  $\tilde{O}(n)$  rounds, and we perform exclude computations from  $n$  sources which takes  $\tilde{O}(n)$  rounds, using Theorem 1. The downcast takes  $n$  rounds and  $O(1)$  congestion per edge, so the computation for all vertices can be scheduled in  $\tilde{O}(n)$  rounds. The computation in lines 4-7 is performed locally at each  $y$ .  $\square$

## 4.2 2-APSiSP Lower Bound

We now prove a lower bound of  $\tilde{\Omega}(n)$  for 2-APSiSP, proving that Algorithm 4 is nearly optimal. A lower bound of  $\tilde{\Omega}(n)$  rounds for computing 2-SiSP for a single pair of vertices in a directed weighted graph was proven in [25]. While this 2-SiSP lower bound trivially applies to computing 2-APSiSP, the lower bound requires the shortest path to have  $\Theta(n)$  hop length, and applies only to directed weighted graphs. In this section, we prove a simpler  $\Omega(n/\log n)$  lower bound for 2-APSiSP even for undirected unweighted graphs, and for graphs where all shortest paths can have constant hop length. The result in Lemma 8 of Section 3.3 with  $k = \Theta(n)$  can also be modified to give a  $\tilde{\Omega}(n)$  lower bound for 2-APSiSP, but only for directed graphs with diameter  $\Omega(\log n)$ .

We first present a lower bound for undirected unweighted APSP using a reduction from set disjointness that we can readily extend to a lower bound for 2-APSiSP. Our construction is inspired by the APSP lower bound in [26], where they use a different communication complexity reduction. Our lower bound of  $\tilde{\Omega}(n)$  also applies to the simpler problem of computing distances between  $n$  given pairs of vertices – note that such a claim cannot hold for  $k = o(n)$  pairs, since we can compute  $k$ -source SSSP in sublinear rounds [24] when  $k$  is sublinear.

**Observation 13.** (*Alternate proof of Observation 1.4 in [26]*) *Given an undirected unweighted graph  $G = (V, E)$ , computing APSP requires  $\tilde{\Omega}(n)$  rounds. This lower bound also applies to the problem of computing the shortest path distance between  $n$  given pairs of vertices.*

*Proof.* Given an  $n$ -bit set disjointness instance  $S_a, S_b$ , we construct a graph with vertices  $a_i, b_i$  for  $1 \leq i \leq n$  and two additional vertices  $a^*$  and  $b^*$ . We include edge  $(a^*, b^*)$  and we add other edges based on the set disjointness input: edge  $(a_i, a^*)$  is present if  $S_a[i] = 1$  and edge  $(b_i, b^*)$  is present if  $S_b[i] = 1$ .

There is a finite shortest path between  $a_i$  and  $b_i$  only if edges  $(a_i, a^*)$  and  $(b_i, b^*)$  exist. Thus,  $d(a_i, b_i) = 3$  if  $S_a[i] = S_b[i] = 1$  and  $d(a_i, b_i) = \infty$  otherwise. In an APSP algorithm, we require node  $b_i$  to know that distances  $d(v, b_i), \forall v \in V$ . Thus, we have the following protocol for set disjointness using an APSP algorithm  $\mathcal{A}$  that takes  $R(n)$  rounds: Alice controls nodes  $a_i, a^*$  and Bob controls nodes  $b_i, b^*$ , and they simulate  $\mathcal{A}$ . Bob can check each  $d(a_i, b_i)$  to test if  $S_a[i], S_b[i]$  are disjoint. The two parties communicate only  $R(n) \cdot \log n$  bits that pass through the cut edge  $(a^*, b^*)$ , so the lower bound for  $n$ -bit set disjointness [31, 5, 21] proves that  $R(n) = \Omega(n/\log n)$ .

Note that our reduction to set disjointness uses only the distances  $d(a_i, b_i)$  for  $1 \leq i \leq n$ , so our reduction also holds for an algorithm that computes shortest path distances between  $n$  given pairs of vertices. Additionally, our lower bound holds for connected graphs with constant undirected diameter since we can connect all vertices to a common path of length  $\geq 4$  without affecting the reduction.  $\square$

**Theorem 5.** (stated in Section 1.2) *Given an undirected unweighted graph  $G = (V, E)$ , computing 2-APSiSP requires  $\tilde{\Omega}(n)$  rounds. The lower bound holds even when  $G$  has constant undirected diameter.*

*Proof.* We extend the lower bound for APSP in Observation 13 to 2-APSiSP. We change the edge  $(a^*, b^*)$  to a path  $a^*-x-b^*$  of length 2. We add two vertices  $c, d$ , and add edges  $(a_i, c)$  and  $(d, b_i)$  for all  $1 \leq i \leq n$ , that are always present irrespective of bits in the set disjointness input. We also add edge  $(c, d)$ . With these modifications,  $d(a_i, b_i) = 3$  using the path through edge  $(c, d)$ . But, the second simple shortest path from  $a_i$  to  $b_i$  must pass through path  $a^*-x-b^*$ , and the argument from Observation 13 applies: We have  $d_2(a_i, b_i) = 4$  if  $S_a[i] = S_b[i] = 1$ , and  $d_2(a_i, b_i) = \infty$  otherwise.

If Alice controls  $a_i, a^*, c$  and Bob controls  $b_i, b^*, d, x$ , there are  $O(1)$  edges connecting Alice's and Bob's vertices. So the lower bound for set disjointness gives an  $\Omega(n/\log n)$  lower bound for 2-APSiSP. Note that the undirected diameter of the graph is  $O(1)$ , so the lower bound holds in undirected unweighted graphs with constant diameter.  $\square$

## 5 Conclusion and Open Problems

In this paper, we present CONGEST upper and lower bounds for DSO and 2-APSiSP. Our bounds for 2-APSiSP are nearly optimal, up to polylog factors, for even undirected unweighted graphs. There still remains a gap between upper and lower bounds for directed weighted DSO, leading to the following open problems:

- For a DSO algorithm that can answer a batch a  $k$  queries in  $O(k + D)$  rounds, we present a lower bound of  $\tilde{\Omega}(n)$  preprocessing rounds, and an upper bound that takes  $\tilde{O}(n^{3/2})$  rounds. Can we bridge the gap between these bounds?
- For a DSO algorithm that takes  $o(k \frac{\sqrt{n}}{\log n} + D)$  rounds, we show a lower bound of  $\tilde{\Omega}(\sqrt{n})$  preprocessing rounds. We show an upper bound that takes  $\tilde{O}(n)$  preprocessing rounds and answers  $k$  queries in  $\tilde{O}(k\sqrt{n} + D)$ . It is unlikely the lower bound can be improved for algorithms taking  $O(k\sqrt{n} + D)$  rounds, as this would require an improved CONGEST SSSP lower bound, but it is an open problem whether the preprocessing can be reduced to sublinear rounds.
- Can we achieve a tradeoff between preprocessing rounds and query response rounds, such as  $\tilde{O}(n^{3/2-c})$  preprocessing rounds to answer  $k$  queries in  $\tilde{O}(kn^c + D)$  rounds, for constant  $0 < c < 1/2$ ? We achieve this for  $c = 0$  and  $c = 1/2$ , but our techniques do not immediately extend to arbitrary  $c$ .



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