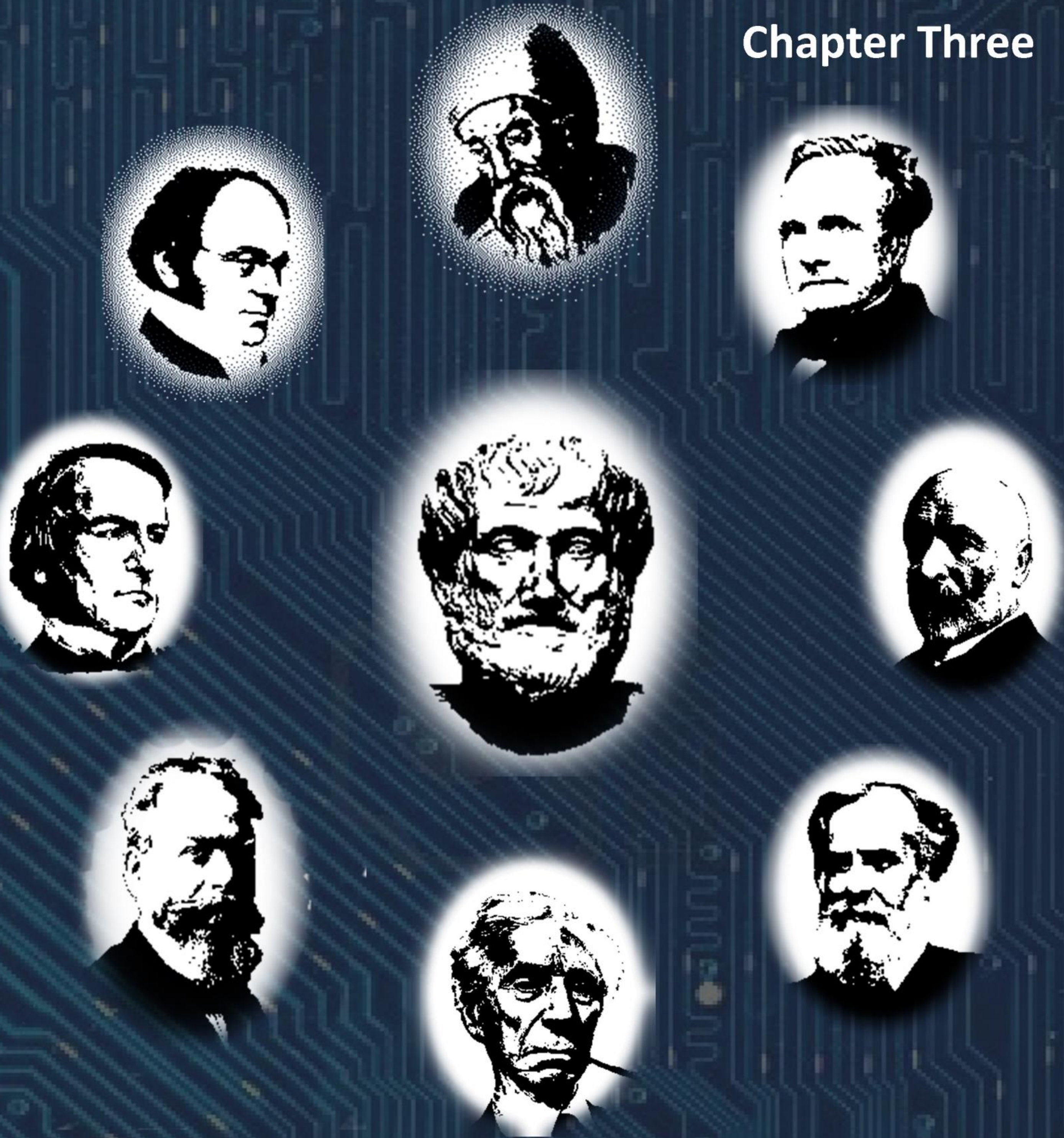
Chapter Three



REASONING

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alan kaylor cline

Boolean Logic Proofs

The Logicians on our cover are:

Euclid (? - ?)

Augustus De Morgan (1806 – 1871) Charles Babbage (1791 – 1871)

George Boole (1815 – 1864) Aristotle (384 BCE – 322 BCE) George Cantor (1845 – 1918)

Gottlob Frege (1848 – 1925) John Venn (1834 – 1923)

Bertand Russell (1872 – 1970)

REASONING

AN INTRODUCTION TO LOGIC, SETS, AND FUNCTIONS

CHAPTER 3
BOOLEAN LOGIC PROOFS

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The University of Texas at Austin

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What Is a Proof?

Introduction

We now have a good deal of experience at using Boolean logic to represent various kinds of things that we might know. But the holy grail, or pot of gold at the end of the rainbow, or jackpot (or whatever metaphor you prefer) isn't representation just for its own sake.

We want to learn new things. We want to derive new claims that must be true given what we already know.

We do this by constructing proofs. In this section we'll learn two techniques for constructing Boolean logic proofs.

Is it the weekend?

No.

How do you know? All I see is that it's Tuesday.

A Proof Is an Argument

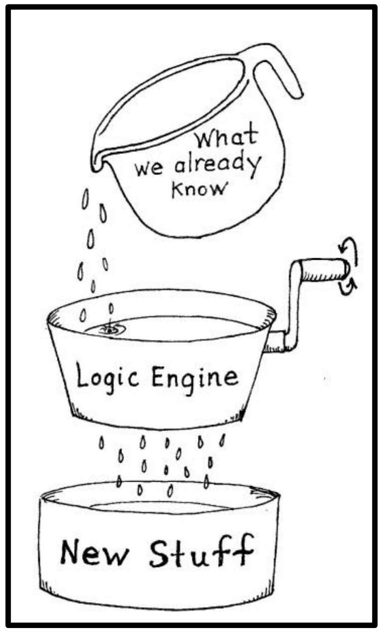
A ***proof*** is an argument that applies one or more:

- sound reasoning methods

to a collection of:

- facts and definitions

to produce a conclusion that must be true whenever the facts are true.



The facts and definitions that we'll use are usually domain-specific. They capture what we know about the particular problem that we are trying to solve.

The reasoning methods, on the other hand, are general. We'll be able to describe them once and then exploit them to discover new things about everyday events, computer circuits, and mathematics. The goal of this course is for you to learn about these reasoning methods so that you can apply them to whatever problems you later encounter.

Wet Sidewalks

Let's do a simple example. We'll give names to the following statements:

R: It's raining.
W: The sidewalks are wet.
S: The sidewalks are slippery.
C: It is important to be careful.

Suppose that we have the following facts:

[1] $R \rightarrow W$ If it's raining then the sidewalks will be wet.
[2] $W \rightarrow S$ If the sidewalks are wet, they will be slippery.
[3] $S \rightarrow C$ If the sidewalks are slippery then it is important to be careful.

At this point, we have no idea whether or not it's important to be careful. But suppose we add one more fact:

[4] R It's raining.

Now, using the kind of reasoning that we do every day, we can say:

- If it's raining then the sidewalks will be wet. But it is raining. So the sidewalks will be wet.
- If the sidewalks are wet, they will be slippery. But they are wet. So they are slippery.
- If the sidewalks are slippery then it is important to be careful. But they are slippery. So it is important to be careful.

We've just constructed a proof that, given what we already knew, it's important to be careful.

We will soon give a name to the inference rule that we just used (three times). It's called ***modus ponens***. It tells us that if we know $p \rightarrow q$ and we know p, then we can conclude q.

We'll be successful at producing proofs when we start out with enough information to enable us to derive useful conclusions. Interestingly, however, our ideas about proofs may help us to solve problems even when we can't actually produce a proof.

Eradicate Ucklufery

Let's give names to the following statements:

V Ucklufery (a very nasty tropical disease) is caused by a virus.
E: We might be able to eradicate ucklufery by developing a vaccine against it.

Suppose that we have the following fact:

[1] $V \rightarrow E$ If ucklufery is caused by a virus, we might be able to eradicate it by developing a vaccine against it.

We know one useful thing about ucklufery. But we're stuck if we try to use it. We don't know enough to reason to any conclusion about whether we should work on a vaccine. But the one thing we do know is a starting point: If we could show that a virus is the culprit, then we would know that we should work on a vaccine. So we actually do know what we should perhaps do next: Attempt to prove that V is true. If it is, then we can apply *modus ponens* to [1] and V to conclude that we should look for a vaccine.

By now you're probably thinking something like, "Okay, I get the proof idea. I've been doing it for years. So what comes next in this course?" The answer is that we are going to formalize the notion of proof so that we'll be sure that we use it correctly. In other words, when we say we have a proof, we'll be sure that the conclusions that we've derived really *must* follow from the facts that we have assumed.

Problems

1. Suppose that we have the following facts:

[1] If the fruit stand sells bananas then they also sell at least one of strawberries or raspberries.
[2] The fruit stand doesn't sell strawberries.

Which of the following claims must be true:

a) The fruit stand doesn't sell bananas.
b) The fruit stand sells raspberries.
c) The fruit stand doesn't sell bananas or does sell raspberries.
d) The fruit stand sells bananas.

2. Suppose that we have the following facts:

[1] If it's Friday or Saturday, the pub will be crowded.
[2] If the pub is crowded, Riley won't go.
[3] Riley is at the pub.

For each of the following claims, indicate whether it must be true, must be false, or could be either true or false:

a) It is Friday.
b) It is Saturday.
c) The pub is crowded.

Premises and Theorems

Every proof that we're going to construct does the same thing. It establishes the truth (validity) of some statement of the following form:

$$(claim_1 \wedge claim_2 \wedge\ claim_3 \wedge \ldots \wedge claim_n) \rightarrow conclusion$$

In other words, it "proves" that, if all the given claims are true, the conclusion must also be true.

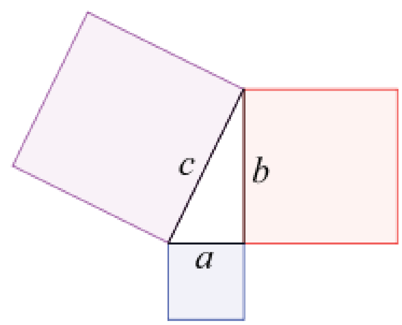
There are at least four common names for what we've just called claims:

- ***premises***
- ***postulates***
- ***hypotheses***
- ***axioms***

When discussing a single argument, taken on its own, without a larger context, it's common to use the word ***premises*** or ***hypotheses***.

Sometimes, however, a single set of claims will be premises to a whole collection of related conclusions (typically called a ***theory***). Then, the premises are usually called the ***axioms*** or ***postulates*** of the theory.

For example, one axiom (postulate) for Euclidean geometry is that, "A straight line segment can be drawn joining any two points". Within that theory, we are not to debate whether this statement is true or false – for the purpose of the theory, we assume it is true. The power of this idea is that a set of carefully chosen axioms may enable us to prove a large body of very useful things. For example, in Euclidean geometry we can prove such things as the Pythagorean Theorem.



Pythagorean Theorem

We will use the four terms premises, postulates, hypotheses and axioms interchangeably.

A ***theorem*** is something that we have proved to be true. So, once we've completed its proof:

- This expression is a theorem:

  $$(claim_1 \wedge claim_2 \wedge\ claim_3 \wedge \ldots \wedge claim_n) \rightarrow conclusion$$

- If we've agreed on a set of axioms and we are building a theory on them, then we'll say (as we just did in the case of the Pythagorean Theorem) simply that this is a theorem:

  conclusion

Big Idea

What's in common here is that a theorem is something we know to be true.

(Premise) $r \to w$	If it's raining then the sidewalks will be wet.
(Premise) $w \to s$	If the sidewalks are wet, they will be slippery.
(Premise) $s \to c$	If the sidewalks are slippery it is important to be careful.
(Premise) r	It's raining.
(Theorem) $((r \to w) \land (w \to s) \land (s \to c) \land (r)) \to c$	If premises true, then it is important to be careful.
(Theorem) c	It is important to be careful.

Once we have proved a statement of the form:

$$(claim_1 \land claim_2 \land \ claim_3 \land \ldots \land claim_n) \to conclusion$$

we can describe what we know in any of these ways:

- The conclusion ***follows from*** the set of claims (or premises or postulates or axioms).
- The set of claims ***logically implies*** the conclusion.
- The set of claims ***entails*** the conclusion.

If you're wondering why, in this course, there seem to be so many ways to say the same thing, all we can tell you is that logic has been around for a long time. A lot of folks have had their hands in the pie. A lot of terms have cropped up. We have to live with them.

Big Idea

Change your premises, watch your conclusions change.

https://www.youtube.com/watch?v=CWpFQBgK4l0

The reasoning techniques that we're about to describe don't say anything about what premises we should start with. The reasoning methods are completely agnostic in that regard. However, and this is a huge "however", that doesn't mean that it doesn't matter what premises we pick. The premises we choose will determine the conclusions that we can draw. Once we choose a set of premises and attempt to produce a proof, we may find ourselves in any one of these situations:

- ***The premises imply a conclusion that we want to draw.*** We will be able to produce a proof. This is what happened in the ***Wet Sidewalks*** example once we added the premise that it is raining.

- ***The premises are too weak; they do not imply the conclusion that we're trying to prove.*** This is what happened in the ***Wet Sidewalks*** example before we added the premise that it is raining. It also happened in the ***Eradicate Ucklufery*** example. When this happens, we typically look for additional premises that we are willing to accept and that would enable us to prove our conclusion.

- ***The premises are wrong. They enable us to prove a conclusion that we believe to be false.*** For example, nothing in logic would have prevented us from starting with the premise, "If it's raining, the sidewalks will be dry." Can you see how this would lead to a conclusion that you'd reject? In this case, we will need to revisit our choice of premises.

- ***The premises are contradictory.*** When this happens, as we'll soon see, it is possible to prove any conclusion. In fact, given contradictory premises, for *any* statement p, it is possible to prove both p and $\neg p$. So, while we'll have proofs, we won't have much useful information.

Problems

1. Suppose that we have the following premises:

[1] If it's Tuesday, we're eating burgers.
[2] If we're eating burgers, of course we're also eating fries.
[3] If we're eating burgers and fries, we're at Tubby's (of course).

We want to prove that we're at Tubby's. Consider the following claims:

I. We can complete the proof with the premises that we've got.
II. We could complete the proof by adding the premise, "It's Tuesday."
III. We could complete the proof by adding the premise, "We're eating burgers."
IV. We could complete the proof by adding the premise, "We're at Tubby's."

Which of the following statements is true:

a) I is true.
b) II is the only claim that is true.
c) III is the only claim that is true.
d) IV is the only claim that is true.
e) II, III, and IV are all true.

2. Suppose that we have the following premises:

[1] If Skip is eating cookies, then Chris is eating popcorn.
[2] If Chris is eating popcorn, then popcorn is a fruit.
[3] If Chris is eating popcorn, then bananas are fruit.
[4] Skip is eating cookies.

Consider the following possible conclusions:

I. Chris is eating popcorn.
II. Popcorn is a fruit.
III. Bananas are fruit.

Which of the following correctly describes the conclusions that we can reach, given our premises:

a) The only one we can derive is [1].
b) The only one we can derive is [2].
c) The only one we can derive is [3].
d) We can't derive any of them.
e) We can derive two or more of them.

Setting Up a Proof

Suppose that we want to show that a set of premises implies a conclusion:

[1] $(premise_1 \wedge premise_2 \wedge\ premise_3 \wedge \ldots \wedge premise_n) \rightarrow conclusion$

In other words, we want to show that there is no circumstance in which the premises are true but the conclusion isn't. Recall that that's equivalent to showing that [1] is a ***tautology***.

So, to construct a proof, we do the following:

1. Choose a set of premises whose truth we are willing to accept.

2. Construct the logical statement that is the conjunction of all of them. That gives us something like:

   $$(premise_1 \wedge premise_2 \wedge\ premise_3 \wedge \ldots \wedge premise_n)$$

3. Construct the logical statement that asserts that such a conjunction implies the desired conclusion. That gives us something like:

   [1] $(premise_1 \wedge premise_2 \wedge\ premise_3 \wedge \ldots \wedge premise_n) \rightarrow conclusion$

4. Show that [1] is a tautology. In other words, that, for any assignment of the values *T* and *F* to the variables in [1], the truth value of [1] is *T*. You might, at this point, argue that we don't actually need the whole column to be *T*. We really only care about the cases where the premises are themselves true. But how are we going to know which rows those are? It's not just when the variables are *T* since (as a trivial example), we could have $\neg p$ as a premise. So we'll insist on a proof that the whole column is *T*. But, in the next section, we'll look at an alternative to truth tables as a proof technique. One of the wins of that alternative (which we'll call, "natural deduction"), is that we won't necessarily have to consider irrelevant combinations of truth values.

Who Drives Me

Let's give names to some basic statements:

J: John must drive me to the store.
M: Mary must drive me to the store.
L: John will be late for work.

Using those statements, we can state our premises:

[1]	$J \vee M$	John or Mary must drive me to the store.
[2]	$J \rightarrow L$	If John drives me to the store, he will be late for work.
[3]	$\neg L$	John cannot be late for work.

The conclusion that we'd like to draw is:

[4] M Mary must drive me to the store.

We want to prove that, if all the premises are true, then the conclusion follows. So we need to form the conjunction of our premises and then set up an implication with that conjunction on the left and the conclusion (i.e., M) on the right. That gives us:

[5] $((J \vee M) \wedge (J \rightarrow L) \wedge (\neg L)) \rightarrow M$

And now we must show that [5] is a tautology. If it is, our premises imply our conclusion.

How shall we prove that we've got a tautology? In the ***Wet Sidewalks*** example, the logical expressions were so simple that we just derived our conclusion informally. But now we have something where it's less obvious how to reason correctly.

We're going to describe two different approaches to constructing sound proofs. The first uses a technique we already know: truth tables.

Then we'll introduce an alternative that we'll call "natural deduction". It corresponds more closely to the way we reason in everyday life. But we'll define it formally so that we're sure that we can't erroneously draw conclusions that don't follow from our premises.

Problems

1. Let's continue with the ***Wet Sidewalks*** example. We'll use the following names for statements:

R:	It's raining.
W:	The sidewalks are wet.
S:	The sidewalks are slippery.
C:	It is important to be careful.
I	It is important to walk rather than run.

Suppose that we have the following premises:

[1]	$R \to W$	If it's raining then the sidewalks will be wet.
[2]	$W \to S$	If the sidewalks are wet, they will be slippery.
[3]	$S \to C$	If the sidewalks are slippery then it is important to be careful.
[4]	R	It's raining.

Consider the following facts that we could add to our list of premises:

I.	I	It is important to walk rather than run.
II.	C	It is important to be careful.
III.	$I \to C$	If it is important to walk rather than run then it is important to be careful.
IV.	$C \to I$	If it is important to be careful then it is important to walk rather than run.

Which of these statements describes how we could add premises to make it possible to conclude that it is important to walk rather than run:

a) I is the only premise that would make the proof possible.
b) III is the only premise that would make the proof possible.
c) IV is the only premise that would make the proof possible.
d) Either II or III would make the proof possible.
e) Either I or IV would make the proof possible.

2. Another extension of the ***Wet Sidewalks*** example. We'll use the following names for statements:

R:	It's raining.
W:	The sidewalks are wet.
S:	The sidewalks are slippery.
C:	It is important to be careful.
U	I should bring an umbrella.

Suppose that we have the following premises:

[1]	$R \to W$	If it's raining then the sidewalks will be wet.
[2]	$W \to S$	If the sidewalks are wet, they will be slippery.
[3]	$S \to C$	If the sidewalks are slippery then it is important to be careful.
[4]	R	It's raining.

Consider the following facts that we could add to our list of premises:

I.	U	I should bring an umbrella.
II.	$S \rightarrow U$	If the sidewalks are slippery then I should bring an umbrella.
III.	$W \rightarrow U$	If the sidewalks are wet then I should bring an umbrella.
IV.	$R \rightarrow U$	If it's raining then I should bring an umbrella.

Which of these statements describes how we could add premises to make it possible to conclude that I should bring an umbrella:

a) I is the only premise that would make the proof possible.
b) III is the only premise that would make the proof possible.
c) IV is the only premise that would make the proof possible.
d) Either I or IV would make the proof possible but none of the others would.
e) Adding any one of the premises would make the proof possible.

3. Another extension of the ***Wet Sidewalks*** example: We'll add the following name for a statement:

Y: It's sunny.

Suppose that we start with the following premises:

[1]	$R \rightarrow W$	If it's raining then the sidewalks will be wet.
[2]	$W \rightarrow S$	If the sidewalks are wet, they will be slippery.
[3]	$S \rightarrow C$	If the sidewalks are slippery then it is important to be careful.
[4]	R	It's raining.
[5]	$W \rightarrow Y$	If the sidewalks are wet, it's sunny. (This one is new.)

Consider the following statements:

[6]	Y	It's sunny.
[7]	$R \wedge Y$	It's raining and it's sunny.
[8]	$\neg R$	It's not raining.

Which of the following correctly describes the new conclusion(s) that we'll now be able to prove, given our premises:

a) Just [6].
b) Just [7].
c) Just [8].
d) Just [6] and [7].
e) All three.

Boolean Logic Proofs Using Truth Tables

Introduction

So far we have used truth tables to tell us when a logical expression is true. It turns out that we can also use them to prove theorems.

Recall that what we need to do in order to prove a theorem is to show that a statement of the following form is a tautology:

$$(premise_1 \wedge premise_2 \wedge\ premise_3 \wedge \ldots \wedge premise_n) \rightarrow conclusion$$

This means that we must show that the conclusion is true for all possible assignments of truth values to its variables. Put another way, it's true in all possible worlds.

So we need to show that every cell of the final column of its truth table contains the value *T*. To do this, we simply build the truth table just as we've been doing.

Let's return to the ***Who Drives Me*** example. We've given these names to the basic statements:

J: John must drive me to the store
M: Mary must drive me to the store
L: John will be late for work.

In terms of these names our premises are:

[1]	$J \vee M$	John or Mary must drive me to the store.
[2]	$J \rightarrow L$	If John drives me to the store, he will be late for work.
[3]	$\neg L$	John cannot be late for work.

And the conclusion we'd like to draw is:

[4]	M	Mary must drive me to the store.

So we must prove that this statement is a tautology:

[5] $((J \vee M) \wedge (J \rightarrow L) \wedge (\neg L)) \rightarrow M$

To do that, we build its truth table (we've omitted a couple of intermediate columns so that the table fits on a page):

J	M	L	$J \vee M$	$J \rightarrow L$	$(J \vee M) \wedge (J \rightarrow L) \wedge (\neg L)$	$((J \vee M) \wedge (J \rightarrow L) \wedge (\neg L)) \rightarrow M$
T	T	T	T	T	F	T
T	T	F	T	F	F	T
T	F	T	T	T	F	T
T	F	F	T	F	F	T
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

Every cell of the final column contains the value *T*. So [5] is a tautology. We have proved that, if the premises are true, then Mary must drive me to the store.

Recall that, when we presented the truth table definition of *implies*, we argued that our definition was a reasonable one:

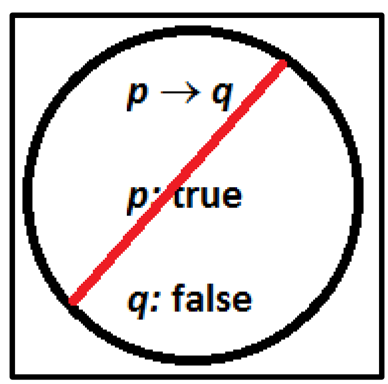
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

You can now see even more clearly why that is so:

- It's the tool we need for proving a theorem: As we just saw, all we have to do to prove that whenever some collection of premises P is true then some conclusion C must also be true is to use the truth table for *implies* to show that $P \rightarrow C$ is a tautology.

- It's the tool we need when it comes time to exploit a theorem to make a new claim: If we know that $p \rightarrow q$ cannot be false (because we have proved it to be a tautology), then we can look at the truth table for *implies* and see that we cannot be in the situation described in line 2. In other words:

  - If p is true then q must be.
  - If p is false then we don't know anything about q. It could be true or false. In other words, this theorem isn't relevant.



Problems

1. Let's return to a famous ***Catch-22*** situation. We'll give names to the following statements:

C: I'm crazy.
M: I've requested a mental health discharge from the Army.
E: I'm eligible for a mental health discharge from the Army.

In Joseph Heller's book, the Army has two rules about this. We can encode them as premises as follows:

[1] $E \rightarrow C \wedge M$ The only way to be eligible is to be crazy and request the discharge.
[2] $M \rightarrow \neg C$ I'm not crazy if I've requested the discharge.

Prove that it's not possible that I'm eligible for a discharge. (And, since we could do this same proof for anyone else, there can never be any of these discharges.)

To do this, we must show that the conjunction of the premises implies the conclusion ($\neg E$). In other words, we must show that the following is a tautology:

$$((E \rightarrow C \wedge M) \wedge (M \rightarrow \neg C)) \rightarrow \neg E$$

Use a truth table to do this.

Not Enough Premises

Recall the key role that premises play. The conclusions that we can draw depend entirely on the assumptions (premises or axioms) that we start with. If we change assumptions we'll get different (possibly fewer, possibly more) theorems. We can now use truth tables to watch this happen.

Let's return to the ***Who Drives Me*** example that we just considered. Let's change one assumption. Suppose that John can be late for work. Now it turns out that we can no longer prove that Mary must drive me to the store. Let's see why not. We drop $\neg L$ as a premise. We now try to use our weaker set of premises to prove this new theorem:

[6] $((J \vee M) \wedge (J \rightarrow L)) \rightarrow M$

Its truth table is:

J	M	L	$J \vee M$	$J \rightarrow L$	$(J \vee M) \wedge (J \rightarrow L)$	$((J \vee M) \wedge (J \rightarrow L)) \rightarrow M$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	F
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

The last column is not all T. So [6] is *not* a tautology. We cannot conclude Mary must drive. It's not that we know that she must not. We simply don't know one way or the other.

Recall the ***Eradicate Ucklufery*** problem that we suggested a while ago. We gave names to the following statements:

V: Ucklufery (a very nasty tropical disease) is caused by a virus.
E: We might be able to eradicate ucklufery by developing a vaccine against it.

We've got one premise:

[1] $V \rightarrow E$ If ucklufery is caused by a virus, we might be able to eradicate it by developing a vaccine against it.

We want to prove that we might be able to eradicate ucklufery by developing a vaccine. So we want to show that the following is a tautology:

$(V \rightarrow E) \rightarrow E$

(We only have one premise, so it's alone on the left of the outer implication.)

Here's the truth table:

V	E	$V \to E$	$(V \to E) \to E$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

Oops. We don't have a tautology. We don't have enough information to be able to conclude that we should work on a vaccine. This example is so simple that it's easy to see what information we lack. If we knew one more thing, namely that ucklufery is caused by a virus, then we'd know that we should work on a vaccine.

Problems

1. Give names to the following statements:

A: It's August.
H: It's hot.
P: There will be a picnic.
R: Randy will make cookies.
S: It's sunny.
W: It's the weekend.

Assume the following premises:

[1] A — It's August.
[2] $W \wedge S \to P \wedge R$ — If it's the weekend and it's sunny, there will be a picnic and Randy will make cookies.
[3] $A \to S \wedge H$ — If it's August, it will be sunny and hot.

We want to prove:

[4] $H \wedge P$ — It will be hot and there will be a picnic.

There aren't enough premises to do this. Which of the following premises, would, if added, enable us to prove the claim? (Hint: If you are stuck, write out the truth table for the tautology that we wish to prove. You'll be able to see which row(s) are not T.)

A $\quad$ H $\quad$ R $\quad$ S $\quad$ W

Wrong Premises

Is it possible to use the logical reasoning engine that we've just described to derive a conclusion that is obviously false? Sure. All we have to do is to choose premises that don't correspond to the world we're trying to reason about.

Let's return to the ***Wet Sidewalks*** example. We've given names to the following statements:

R: It's raining.
W: The sidewalks are wet.
S: The sidewalks are slippery.
C: It is important to be careful.

Here are the premises that we've been using:

[1] $R \rightarrow W$ If it's raining then the sidewalks will be wet.
[2] $W \rightarrow S$ If the sidewalks are wet, they will be slippery.
[3] $S \rightarrow C$ If the sidewalks are slippery then it is important to be careful.
[4] R It's raining.

We can easily use a truth table to prove any of the conclusions that we derived earlier using everyday logic (actually we used the logical rule *modus ponens* that we'll soon define formally). Let's prove that the sidewalks will be wet:

R	W	$R \rightarrow W$	$(R \wedge (R \rightarrow W)$	$(R \wedge (R \rightarrow W)) \rightarrow W$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Success. The final column is all T's.

But now let's change our premises. In particular, let's change the first one so that it asserts that, if it's raining, the sidewalks will be dry. Let D stand for the assertion that the sidewalks are dry. Now we can easily prove that, if it's raining the sidewalks will be dry:

R	D	$R \rightarrow D$	$R \wedge (R \rightarrow D)$	$(R \wedge (R \rightarrow W)) \rightarrow D$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Oops. We've just proved something that's clearly nonsense. This is typically what happens if we don't choose our premises wisely.

Big Idea

We must separate the validity of an argument from the reasonableness of its premises.

Problems

1. Give names to the following statements:

B: Bananas grow here.
H: It is hot here.
M: Monkeys live here.
R: Reindeer live here.
S: There is snow here.

Assume the following premises:

[1]	H	It is hot here.
[2]	$\neg(R \wedge B)$	There can't be both reindeer and bananas.
[3]	$H \rightarrow R$	If it's hot, there will be reindeer.
[4]	$B \rightarrow M$	If there are bananas, there will be monkeys.
[5]	$R \rightarrow S$	If there are reindeer, there will be snow.

Using these premises, it is straightforward to prove (try it yourself) that:

[6] $H \wedge S$ It is hot and there is snow.

But [6] is nonsense. Assume that we are certain that it is hot.

(Part 1) Which of the other premises is far from being true in the real world and has the property that, if we simply removed it, we'd no longer be able to generate nonsense such as [6]?

(Part 2) Which of the following premises would be a good replacement for the wrong one above? (In other words, which would do a good job of describing the world in which we live?)

a) $\neg(R \wedge S)$
b) $S \rightarrow \neg H$
c) $H \rightarrow S$
d) $S \rightarrow B$
e) $B \vee M$

Contradictory Premises

What happens if we choose premises that aren't just wrong (i.e., they don't correctly describe the world)? What happens if they actually contradict each other? It turns out that, if we do that, in even one case, we'll be able to prove any conclusion we can come up with.

To see why this is so, let's abstract away from any particular premises that we might choose. Let's consider:

- an arbitrary premise we'll simply call p, and
- some arbitrary conclusion we'll call q.

Now suppose that we add the premise $\neg p$. We know that it's not possible for both p and $\neg p$ to be true. (Recall Aristotle's Principle of Non-Contradiction.) But suppose that we claim that they are. What happens? Let's now try to prove q from our two (contradictory) premises. To do this we must show that this is a tautology:

$$(p \wedge \neg p) \rightarrow q$$

Here's the truth table:

p	q	$\neg p$	$p \wedge \neg p$	$(p \wedge \neg p) \rightarrow q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

We've proved q. And we've done it without any appeal to what p and q actually are and without any premises that connect p and q in any way. Whatever p is, $p \wedge \neg p$ is false. $F \rightarrow q$ is always true (from the definition of implies). So q is true.

For example if we assert both of the following claims:

- The moon is made of green cheese.
- The moon is not made of green cheese.

Then we can prove any of the following:

- Elephants can fly.
- Elephants cannot fly.
- The king of France is a unicorn.

Notice why this is. Look at the next to the last column of the truth table. It's all F's. Recall the definition of the expression $p \rightarrow q$. It's true whenever p is false (as well as whenever q is true). So if the first operand (the part before the $\rightarrow$) is false, the entire expression will always be true.

Big Idea

With contradictory premises we can prove anything. Beware.

Of course, in toy examples like this one, it's easy to see that we've added contradictory premises and so our logic engine is of no use in attempting to determine truth. There's a much more serious problem, however, if we have thousands, hundreds of thousands, or even millions of premises. This can happen when Boolean logic is used to solve real, practical problems like the design of computer circuits. In those cases, engineers must be very careful and they must exploit powerful design tools to guarantee the premises are not contradictory.

Problems

1. Let's give names to the following statements:

S: I should study Spanish today.
G: I should study Government today.
H: I will stay at home today.

Assert the following premises:

[1]	$H \rightarrow (S \vee G)$	If I stay home today I should study Spanish or Government.
[2]	$\neg G$	I am not going to study Government today.
[3]	H	I will stay at home today.

We wish to prove that I should study Spanish today.

(Part 1) We need to show that the claim that the premises imply the conclusion is a tautology. Which of the following statements is that claim:

a) $(H \rightarrow (S \vee G)) \wedge \neg G \wedge H \wedge S$
b) $((H \rightarrow (S \vee G)) \wedge \neg G \wedge H) \rightarrow S$
c) $((H \rightarrow (S \vee G)) \vee \neg G \vee H) \rightarrow S$
d) $((H \rightarrow (S \vee G)) \wedge \neg G) \rightarrow S$

(Part 2) We will use a truth table to prove this claim. As one builds a truth table, there are sometimes choices about what intermediate expressions to make explicit columns for. But some expressions would be useless. Which of these would get us nowhere in building the truth table that we need?

a. $H \rightarrow (S \vee G)$
b. $\neg G \vee H$
c. $S \vee G$
d. $\neg G \wedge H \rightarrow S$

(Part 3) Use the Truth Table app to build the table that proves the claim.

2. Let's give names to the following statements:

C: I will serve cake at my party.
P: I will serve pie at my party.
B: I will buy a cake.
I: I will buy a pie.
M: I have money.

Assert the following premises:

[1]	$C \vee P$	I will serve cake or pie at my party.
[2]	$C \rightarrow B$	If I serve cake, I will buy a cake.
[3]	$B \rightarrow M$	If I buy a cake, I have money.
[4]	$\neg M$	I have no money.

We wish to prove that I will serve pie at my party.

(Part 1) We need to show that the claim that the premises imply the conclusion is a tautology. Which of the following statements is that claim:

a) $((C \vee P) \wedge B_P \wedge \neg M) \rightarrow P$
b) $((C \vee P) \wedge (C \rightarrow B) \wedge (B \rightarrow M) \wedge \neg M \wedge \neg B) \rightarrow P$
c) $P \rightarrow \neg C$
d) $((C \vee P) \wedge (C \rightarrow B) \wedge (B \rightarrow M) \wedge \neg M) \rightarrow P$

(Part 2) We will use a truth table to prove this claim. We know that the number of rows in our truth table grows as the number of propositional variables grows. We've defined five variables in this problem. But we don't have to enter into the truth table any that aren't involved in the proof. How many of the variables that we've defined do we actually need to use to do this proof?

a) 2
b) 3
c) 4
d) 5

(Part 3) How many rows will the truth table have?

a) 4
b) 8
c) 12
d) 16
e) 20

(Part 4) Show the truth table that proves our claim.

3. Let's give names to the following statements:

N: It's raining.
C: It's clear.
U: Unicorns are purple.

Assert the following premises:

[1]	N	It's raining.
[2]	$C \rightarrow \neg N$	If it's clear, it's not raining.
[3]	C	It's clear.

(Part 1) Using these premises, we wish to prove that unicorns are purple. Which of the following statements, if it's a tautology, proves the claim:

a) $(N \wedge (C \rightarrow \neg N) \wedge C) \rightarrow U$
b) $(N \vee C) \rightarrow U$
c) $(N \wedge (C \rightarrow \neg N) \wedge C) \rightarrow (U \vee \neg U)$
d) $(C \rightarrow \neg N) \vee N) \rightarrow U$

(Part 2) Write out the truth table. Can you prove the claim that unicorns are purple?

(Part 3) Now, on the other hand, we wish to prove that unicorns are not purple. Which of the following statements, if it's a tautology, proves the claim:

a) $(N \vee C) \rightarrow \neg U$
b) $(N \wedge (C \rightarrow \neg N) \wedge C) \rightarrow (U \vee \neg U)$
c) $(N \wedge (C \rightarrow \neg N) \wedge C) \rightarrow \neg U$
d) $((C \rightarrow \neg N) \vee N) \rightarrow \neg U$

(Part 4) Write out the truth table. Can you prove the claim that unicorns are not purple?

(Part 5) Suppose that we want to delete premises until it's no longer possible to prove that unicorns are purple. (After all, we don't actually have any premises that say anything about unicorns.) Which of the following deletions will do the job:

a) Deleting N is the only thing that will accomplish the task.
b) Deleting C is the only thing that will accomplish the task.
c) Deleting $C \rightarrow \neg N$ is the only thing that will accomplish the task.
d) Deleting any one of the premises will accomplish the task.

Proving Other Kinds of Claims

So far, we've used truth tables to prove claims of the form:

$$(Premise_1 \wedge Premise_2 \wedge \ldots Premise_n) \rightarrow Conclusion$$

We do that by showing that such a claim is a tautology.

But we can also use truth tables to prove the correctness of other kinds of claims (again by showing that they are tautologies). For example, we might want to prove that two logical expressions are equivalent.

Prove that these two logical expressions are equivalent (i.e., for any assignment of truth values to the propositions, either both expressions are true or both are false):

$P \wedge \neg Q$ $\qquad\qquad$ $\neg(P \rightarrow Q)$

To do this, we will prove that the following claim is a tautology:

$(P \wedge \neg Q) \quad \equiv \quad (\neg(P \rightarrow Q))$

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$	$(P \wedge \neg Q) \equiv (\neg(P \rightarrow Q))$
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T

Problems

1. Prove that $((p \wedge q) \rightarrow r) \equiv (p \rightarrow (q \rightarrow r))$. Use the truth table app.

Theorem upon Theorem

Suppose that we want to prove something even a little bit less trivial than the examples that we've done so far. For example, what if we need 10 variables. Then we know that the truth table that we'd have to build would have 2^{10} or 1024 rows. Ouch. And things get much worse very quickly. If we needed 50 variables, we'd need 2^{50} or 1,125,899,906,842,624 rows. Clearly we need some new techniques.

Here are two ideas. We're going to see that both of them are powerful and, in fact, we'll want to combine them:

- Prove smaller theorems using smaller collections of variables. Then treat those theorems like premises. (This is okay because we know that they must be true.) Build on them until we prove the final theorem that we care about.

- Come up with a new technique that lets us focus on the specific ways in which the premises connect to each other. In most cases, this effectively lets us ignore many (most) of the possible truth value combinations for the variables.

We'll consider the first of these techniques here. Then, in the next section, we'll take the second approach and develop a whole new way to construct proofs.

Let's return to the ***Who Drives Me*** example.

We'll use these propositional variable names for the basic statements:

J:	John must drive me to the store.
M:	Mary must drive me to the store.
L:	John will be late for work.
G:	Mary must buy gas. (new)
D:	Mary must have money. (new)

Assume these premises:

[1]	$J \vee M$	John or Mary must drive me to the store.	
[2]	$J \to L$	If John drives me to the store, he will be late for work.	
[3]	$\neg L$	John cannot be late for work.	
[4]	$M \to G$	If Mary must drive me to the store, she must buy gas.	(new)
[5]	$G \to D$	If Mary must buy gas, she must have money.	(new)

And we've already proven:

[6]	M	Mary must drive me to the store.

Suppose that we want to prove:

[7] D Mary must have money.

We could start from scratch and prove:

[8] $((J \vee M) \wedge (J \rightarrow L) \wedge (\neg L) \wedge (M \rightarrow G) \wedge (G \rightarrow D)) \rightarrow D$

Notice that there are now five premises [1] – [5] *and*ed together on the left of the top level *implies*. More importantly, there are now five propositional variables. So, if we build a truth table, from scratch, to prove our claim, we'll need 32 rows. We can do it, but it's extremely tedious.

But suppose that, instead of premises [1] – [3], we just use [6], which we've already derived from them. Then we can ignore the variables *J* and *L* too. We can build a truth table proof that corresponds to the everyday reasoning chain:

- Mary must drive me to the store. So she must buy gas. So she must have money.

So now the truth table that we have to build is the one that shows that this claim is a tautology:

[9] $(M \wedge (M \rightarrow G) \wedge (G \rightarrow D)) \rightarrow D$

Since only three variables are involved, we'll just need to build an eight-row truth table.

Big Idea

Divide and conquer: Break complex proofs into smaller, more manageable pieces.

Boolean Identities

Introduction

The truth table is a universal tool for working with Boolean logic expressions. In principle, it's all we need. In practice, however, it gets cumbersome quickly. We've already seen how that can happen even we're still dealing with trivial ideas. So we're going to want something else.

In particular, we're going to want a set of techniques for manipulating logical expressions to make them more useful. But we'll need to guarantee that the manipulations that we do cannot affect truth value.

Recall that there's a natural analogy between algebraic (arithmetic) expressions and logical ones. We can continue that analogy here.

> So, for example, in algebra we have that these two expressions are equivalent:
>
> $2x^3 + 17 + 5x^2$ $\qquad\qquad$ $2x^3 + 5x^2 + 17$
>
> We can transform the first into the second, without changing its value, by exploiting the fact that addition is commutative (i.e., it doesn't matter what order we do the additions in).

> Similarly, in algebra, we have that these two expressions are equivalent:
>
> $(a + b)\ (c + d)$ $\qquad\qquad$ $ac + ad + bc + bd$
>
> This time, we know that we can transform the first into the second, without changing its value, by exploiting a distributivity property: multiplication distributes over addition.

In algebra, we sometimes use these identities to transform an expression into another one that happens to be more useful for some particular purpose. We also use them to simplify expressions.

> For example, these two expressions are equivalent:
>
> $(a + b)\ (c + d) - bc$ $\qquad\qquad$ $ac + ad + bd$
>
> We get the second one by first applying the distributive property and then cancelling out the *bc* and –*bc* terms.

Problems

1. Indicate, for each of the following pairs of arithmetic expressions, whether or not they are equivalent:

a)	$(a + b) * c$	$ac + bc$
b)	$x * (y * z)$	$(z * x) * y$
c)	$-(b + c)$	$-b + \mathrm{c}$
d)	$x + y - z$	$x - z + y$

A List of Identities

In Boolean logic, we also have a set of identities that enable us to transform expressions without changing their values (in this case, their truth values). Some of them are analogs of the arithmetic identities. For example, both *or* and *and* are commutative. Some will be new.

Here's a list of identities that we'll find most useful. The way to prove them is to use truth tables to show that the truth values of both sides are the same. We'll prove the first one. We suggest that you prove at least a few more of them. It will give you practice using truth tables and you'll find yourself proving your first useful theorems.

Notice that each of these identities has the form:

$$expression_1 \equiv expression_2$$

Recall that the symbol ≡ means *is equivalent to*. What we're claiming, in our statement of each of these identities, is that the two sides of the equivalence statement have the same truth values. And we're claiming that this holds for all propositions (*p*, *q*, *r*, or even ones that themselves contain Boolean operators).

Double Negation

Double Negation: $p \equiv \neg(\neg p)$.

In other words, in logic, two negatives cancel each other out.

To prove this claim, we build a truth table that has one column for the left hand side of the equivalence and one column for the right hand side. Then we build a final column that corresponds to the claim that those two are the same. We have a proof if that final column contains all *T*'s. (Notice that there may be additional working columns as well. We don't care what their values are as long as they lead to the two critical columns being the same.) We've already seen the first three columns of the truth table that we need. We showed them when we pointed out that applying *not* twice gets us back where we started. Here's the whole table that we need to build to prove the double negation identity:

p	$\neg p$	$\neg(\neg p)$	$p \equiv \neg(\neg p)$.
T	*F*	*T*	*T*
F	*T*	*F*	*T*

We've highlighted the two columns that correspond to the two sides of the equivalence. Notice that they are identical. So the final column contains all *T*'s.

English Aside

Double negation shows up much more often in English than you might imagine. One reason it happens is that many words other than "not" actually mean *not*.

Let *P* be the statement: "I will go to the game."

Then: "No way will I miss that game," is $\neg\neg P$,

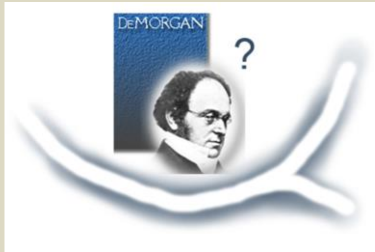
Since: "miss" means $\neg P$.

So you know *P*: "I will go to the game."

We should remind you though that, in some dialects, double negation means the same thing as single negation. For example:

"Don't nobody love me."

Nifty Aside



The Fork in the Road

A weary traveler approaches a fork in the road. Not knowing which way to go, he decides he should ask the one native he sees. But he knows that there's a problem. This is the land of the liars and the truth-tellers. Everyone is either a liar (who lies absolutely all the time) or a truth-teller (who tells the truth absolutely all the time. Unfortunately, these folks don't wear affinity tee shirts. There's no way to tell, when you're talking to someone, which camp he's in. And there's one more thing: It's well known that strangers are allowed to ask only a single question before they must move on.

Fortunately, our traveler is a logician. He asks a single question and, without knowing whether he's talking to a liar or a truth-teller, gets the answer that he needs. What question does he ask?

This puzzle is fun. See if you can work it out.

Problems

1. Suppose that we take as a premise:

   It's not impossible that Riley will win.

   Is it possible that Riley will win?

More Identities

We present the rest of the identities without proof. You should try a few of the proofs yourself.

Equivalence: $(p \equiv q) \equiv (p \to q) \wedge (q \to p)$

In other words, two statements are equivalent just in case each implies the other.

This rule suggests a way to go about proving that two statements are equivalent. Construct two proofs, one for each direction of the implication.

Idempotence:

$$(p \wedge p) \equiv p$$
$$(p \vee p) \equiv p$$

"Idempotence" is a big word that just means, "has the same power as itself". Despite the word, these laws probably seem completely obvious. If we *or* (or *and*) something with itself, we just get back what we started with. These laws, while seemingly trivial, can be very useful when we need to simplify logical expressions.

De Morgan's Laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

These two laws are named for the important 19th century logician Augustus De Morgan. They are useful in simplifying expressions because they attach *not*s to smaller units, thus making them easier to work with. When we use De Morgan's laws, we sometimes describe what we're doing as, "pushing *not* through *and*" or, "pushing *not* through *or*."

Read De Morgan's laws carefully: When we apply them, *or*s become *and*s and *and*s become *or*s.

English aside

Consider the English sentence, "I won't touch beets *or* okra." Do you agree that this means both no beets *and* no okra for me?

Let's give names to these sentences:

B: I will touch beets.
O: I will touch okra.

We can write our sentence in logic as:

$\neg(B \vee O)$

Using the second of De Morgan's laws, we can rewrite that as:

$\neg B \wedge \neg O$

A straightforward translation of this into English gives us:

I won't touch beets *and* I won't touch okra.

English aside

Here's another one, this time going from *and* to *or*: Consider the sentence, "You can't have both cake *and* pie." Do you agree that this means that you're not getting pie *or* you're not getting cake?

Let's give names to these sentences:

C: You can have cake.
P: You can have pie.

We can write our sentence in logic as:

$\neg(C \wedge P)$

Using the first of De Morgan's laws, we can rewrite that as:

$\neg C \vee \neg P$

In other words, either you're not getting cake or you're not getting pie.

But, the ***Cooperative Principle*** may come into play:

This sentence doesn't actually say that you're even going to get one of them. You might be both cake and pie deprived. But most of us agree that if that were the case, it would be uncooperative to say this. The mean person should have said, "You're not getting cake *or* pie."

Commutativity of *or* and *and*: $(p \vee q) \equiv (q \vee p)$.
$(p \wedge q) \equiv (q \wedge p)$.

Like addition and multiplication, both *or* and *and* are commutative.

Notice that *implies* isn't on this list; it isn't commutative.

Associativity of *or* and *and*: $(p \vee (q \vee r)) \equiv ((p \vee q) \vee r)$.
$(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$.

Again, like addition and multiplication, both *or* and *and* are associative.

Notice again that *implies* isn't on this list. That's because it isn't associative.

Distributivity of *and* and *or*: $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$.
$(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$.

Just as addition distributes over multiplication, *and* distributes over *or* and vice versa.

Conditional Disjunction: $(p \rightarrow q) \equiv (\neg p \vee q)$.

Sometimes it's useful to think of this as an alternative way of writing $(p \rightarrow q)$. It avoids the use of implies, which may seem confusing.

English aside

We sometimes use this alternative in English. An equivalent fortune would have been, "If you display your treasures, people will become envious."

Lots of people think that the word "implies" connotes a causal relationship (in addition to the purely logical one that we're working with). If you, or your audience, is inclined to make this mistake, it reduces confusion if you use Conditional Disjunction to avoid using the word, "implies".

Contrapositive: $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$.

The intuition here is that, if p implies q but we observe not q, then we know that p must be false. For example, if rain implies wet sidewalks and we observe dry sidewalks, we can conclude that there isn't rain.

Problems

1. Consider the following sentence:

Either Alabama or Florida State will not play in the national title game.

Give names to the following statements:

A: Alabama will play in the national title game.
F: Florida State will play in the national title game.

Consider the following formulas:

I. $\neg A \vee \neg F$
II. $\neg(\neg A \vee \neg F)$
III. $\neg(A \wedge F)$

Which of them represent(s) the intended meaning of our sentence:

a) Just I.
b) Just II.
c) Just III.
d) Just I and III.
e) All three.

2. Consider the following sentence: If you come home early, you'll be disappointed.

Give names to the following statements:

E: Come home early.
D: Disappointed.

Consider the following formulas:

I. $E \rightarrow D$
II. $D \vee \neg E$
III. $\neg D \rightarrow \neg E$

Which of them represent(s) the intended meaning of our sentence:

a) Just I.
b) Just II.
c) Just III.
d) Just I and III.
e) All three.

3. Suppose that we know that $(\neg R \wedge S) \wedge (\neg P \rightarrow R) \wedge (Q \vee R)$ is true. For each of the variables, mark True if it MUST be true, False if it must be false, or Either if it could be either true or false.

a) P
b) Q
c) R
d) S

A Nonidentity – Converse

Notice that our last identity, the contrapositive, tells us that if we know that p implies q, then there is also a way to reason from something about q to something about p. But it's very specific. If we know *not* q, then we also know *not* p. But what if we know q? Can we conclude p? In other words, can we reverse $(p \rightarrow q)$ and derive $(q \rightarrow p)$? The answer to this question is an emphatic *no*.

https://www.youtube.com/watch?v=tSu0Pg0nWJY

A simple real world example shows us why we'd be upset if our logic let us do that. Suppose that we have:

[1] Rain implies wet sidewalks.

Should we be certain that:

[2] Wet sidewalks imply rain.

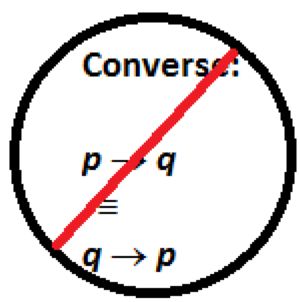
Of course not. Maybe there's rain. But maybe not. Someone could just have washed the sidewalks. Or someone's sprinkler system could be on. Or the water main could have broken.

Define the ***converse*** of the expression $(p \rightarrow q)$ to be the expression $(q \rightarrow p)$. Then we have:

Converse does not follow: $((p \rightarrow q) \equiv (q \rightarrow p))$ is *not* a tautology.

We prove this with a truth table as well:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \equiv (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T



Notice that the final column is not all T's. What we see is that there are some cases where the two expressions have the same truth value. But that's not guaranteed. There are other cases where they are different.

We're making sort of a big deal about this one because inferring the converse is one of the most common logical errors that people make.

Problems

1. In each of the following problems, assume [1] as a premise. Then consider [2]. Suppose that we accepted Converse as an identity (in addition to the actual identities listed above). Mark Follows if [2] would then be derivable from [1]. Mark Does Not Follow if [2] would still not follow from [1], even if Converse were an identity.

a) [1] $Speeding \rightarrow Ticket$
   [2] $Ticket \rightarrow Speeding$

b) [1] $Hungry \vee LunchTime \rightarrow Eat$
   [2] $Eat \rightarrow Hungry$

c) [1] $HighMountain \rightarrow Snow \wedge \neg Trees$
   [2] $Snow \rightarrow HighMountain$

Necessary and Sufficient Conditions

Recall the truth table for $\rightarrow$ (*implies*):

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

https://www.youtube.com/watch?v=QtMFyTV8jfg

Suppose that I say, "p implies q". Then I've said that p is a ***sufficient condition*** for q. In other words, I'm claiming that, if you wanted to back up the claim that q must be true, it would be sufficient to show that p is true. Said another way, p cannot be true without q also being true.

Notice also that, using the Contrapositive identity, we have: $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$

So, if $p \rightarrow q$ (i.e., p is a sufficient condition for q), then $\neg q \rightarrow \neg p$ (i.e., $\neg q$ is a sufficient condition for $\neg p$).

But now suppose that I want to claim that p is a ***necessary condition*** for q. In other words, q cannot be true unless p also is. (Note that, even if p is true, q might not be. I just know that, for sure, if p isn't true then q isn't.) I could state this claim as, "q only if p".

Here's a truth table for *q only if p*, which is true except in the case in which q is true but p isn't:

p	q	*q only if p*
T	T	T
T	F	T
F	T	F
F	F	T

There's an important relationship between *implies* and *only if*, which we can see from the following truth table. Column 3 repeats the values for "$p \rightarrow q$". In column 4, we've repeated the truth table for *only if* except that we've swapped p and q, so it shows the values for "p only if q" (which is false only in case p is true but q is false):

p	q	$p \rightarrow q$	*p only if q*
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Notice that columns 3 and 4 of this table are the same. Putting all this together, we see that, if we know:

$p \rightarrow q$

then we have:

- p is a sufficient condition for q,
- $\neg q$ is a sufficient condition for $\neg p$ (using Contrapositive),
- q is a necessary condition for p (note again that q and p are reversed here), and
- $\neg p$ is a necessary condition for $\neg q$ (from the truth table for $p \rightarrow q$. The only row where $p \rightarrow q$ is true and q is false is row 4, in which p is also false).

Suppose we claim, "If it rains, the sidewalks will be wet". Then we could also say:

- "Rain is a sufficient condition for wet sidewalks." (In other words, to argue wet sidewalks, it is enough to argue that it is raining.
- "Dry (non-wet) sidewalks are a sufficient condition for not rain."
- "Wet sidewalks are a necessary condition for rain." (In other words, there can't be rain without there being wet sidewalks.
- "No rain is a necessary condition for dry sidewalks."

We'll soon see, when we look at techniques for constructing Boolean logic proofs, that the two ways to think of what *implies* tells us about sufficient conditions turn out to enable us to use the statement "$p \rightarrow q$" both as a way to prove q (if we happen to know p) and as a way to prove $\neg p$ (if we happen to know $\neg q$).

Notice one last thing. Given $p \rightarrow q$ (and thus that p is a sufficient condition for q), there is one thing that we cannot conclude:

- q is a sufficient condition for p. This would have to be true if the converse of $p \rightarrow q$ (i.e., $q \rightarrow p$) were guaranteed to be true. But it isn't. It might turn out that q is a sufficient condition for p, but we would have establish the truth of that claim separately.

Now suppose that I want to claim that p is a ***necessary and sufficient condition*** for q. In other words the truth of p guarantees the truth of q and the truth of q guarantees the truth of p. Let's make the truth table for that:

p	q	$p \rightarrow q$	p *only if* q	$(p \rightarrow q) \wedge (q$ *only if* $p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

The title of the last column of this table is $(p \rightarrow q) \wedge (q$ *only if* $p)$. But this relationship, namely that p is a necessary and sufficient condition for q, is important enough that it needs its own pronounceable name. When this relationship holds, we'll say:

p ***if and only if*** q, which can be shortened to: p ***iff*** q.

One of the most important uses of *iff* is in definitions. It's common, in English, to write definitions using just *implies*. But what is in fact meant is *iff*. We'll need to be careful about this.

Consider these examples:

1. "A triangle is a right triangle if one of its angles is 90°."
2. "A logical operator is a binary operator if it takes two operands."
3. "A color is 'warm' if it is in the red through yellow part of the spectrum."

Look again at the last column of what we are now calling the *iff* truth table. Notice that it's identical to the last column of the truth table for *is equivalent to*:

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

So now we see that saying, "*p* is equivalent to *q*", is the same as saying, "*p* is a necessary and sufficient condition for *q*" (or that, "*q* is a necessary and sufficient condition for *p*") or that, "*p* if and only if *q*" (or "*q* if and only if *p*").

Problems

In all of these problems, assume the real world in which we live.

1. Let *G* stand for the claim, "The grass is wet." Now consider the following additional claims:

I. It's raining.
II. The sprinkler is on.
III. It's sunny.

(Part 1) Which one or more of the numbered claims is a sufficient condition for *G*?

(Part 2) Which one or more of the numbered claims is a necessary condition for *G*?

2. Let *G* stand for the claim, "I can graduate from the University of Texas." Now consider the following additional claims:

I. I've been admitted to UT.
II. I've made some tuition payments to UT.
III. I've completed all of UT's graduation requirements.

(Part 1) Which one or more of the numbered claims is a sufficient condition for *G*?

(Part 2) Which one or more of the numbered claims is a necessary condition for *G*?

3. Consider the question, "The lamp is plugged in, so it will work, right?" Which of the following is a correct answer to the question:

a) Yes. Being plugged in is a sufficient condition for a lamp to work (although I should point out that it isn't necessary).
b) Yes. Being plugged in is a sufficient (and also, as it turns out, a necessary) condition for getting a lamp to work.
c) No. Being plugged in, while it is necessary, it is not a sufficient condition for a lamp to work.

4. Consider the following statements:

I. A number is even if it is evenly divisible by 2.
II. A costume is cool if it involves a superhero.
III. A song is a hit if it has sold more than 1,000,000 copies.

In which one or more of them did we actually mean "iff", even though we just said "if"?

5. Consider the following statements:

I. A novel is Victorian if it was written in English between 1837 and 1901.
II. A joke is bad if everyone groans loudly when they hear it.
III. Food smells bad if it's moldy.

In which one or more of them did we actually mean "iff", even though we wrote "if":

Computation

Once we've proved all of these identities, we can use them to transform and simplify logical expressions. If necessary, we can apply several of them, one step at a time.

As we do that, we may find that we want to compute logical values. Appealing to our analogy with algebra, we have that:

$x + 5 - 5$	can be simplified to:	x
$5 + 7$	can be simplified to:	12

In logic, we compute with the truth values T and F. It is straightforward to prove all of these equivalences with truth tables:

• $p \vee \neg p$	$\equiv$	T	$\neg p \vee p$	$\equiv$	T
• $p \wedge \neg p$	$\equiv$	F	$\neg p \wedge p$	$\equiv$	F
• $p \vee T$	$\equiv$	T	$T \vee p$	$\equiv$	T
• $p \vee F$	$\equiv$	p	$F \vee p$	$\equiv$	p
• $p \wedge T$	$\equiv$	p	$T \wedge p$	$\equiv$	p
• $p \wedge F$	$\equiv$	F	$F \wedge p$	$\equiv$	F

When we use one of these facts, we'll label that step with the justification **Computation**.

Suppose that we are presented with the following logical expression (maybe as the solution to some circuit design problem):

$$p \wedge \neg(\neg s \vee p)$$

We want to try to simplify it before we use it for something else. We'll show here a set of steps that do that. On each line (except the first), we'll indicate the identity that we used in moving to it from the equivalent logical expression above it. We'll also underline the part of the input expression to which the identity applied.

[1]	$p \wedge \underline{\neg(\neg s \vee p)}$	
[2]	$p \wedge (\underline{\neg\neg s} \wedge \neg p)$	De Morgan
[3]	$p \wedge (\underline{s \wedge \neg p})$	Double Negation
[4]	$\underline{p \wedge (\neg p \wedge s)}$	Commutativity of *and*
[5]	$(\underline{p \wedge \neg p}) \wedge s$	Associativity of *and*
[6]	$\underline{F \wedge s}$	Computation
[7]	F	Computation

Problems

The problems in this section are simple and give you a chance to practice working with individual identities.

1. Given the statement: $\neg(p \wedge \neg q) \vee r$

Which of the following alternative statements is equivalent to the one we've been given:

I. $(\neg p \vee q) \vee r$
II. $(p \vee \neg q) \vee r$
III. $(\neg p \vee q) \vee \neg r$

a) Just I.
b) Just II.
c) Just III.
d) Just I and II.
e) Just II and III.

2. Given the statement: $p \rightarrow (q \vee r)$

Which of the following alternative statements is equivalent to the one we've been given:

I. $(\neg p \vee q) \vee r$
II. $(q \vee r) \rightarrow p$
III. $(p \rightarrow q) \vee r$

a) Just I.
b) Just II.
c) Just III.
d) Just I and II.
e) Just I and III.

3. Prove that *implies* isn't associative. To do this, you need to show that:

$$(p \rightarrow q) \rightarrow r$$

is *not* logically equivalent to:

$$p \rightarrow (q \rightarrow r)$$

Use a truth table to show that this claim is *not* a tautology:

$$((p \rightarrow q) \rightarrow r) \equiv (p \rightarrow (q \rightarrow r))$$

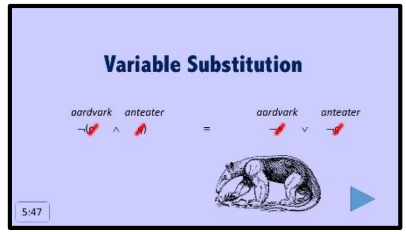
The p’s and q’s are Placeholders

The identities that we have just defined (and inference rules that we are about to define) are sound ways of deriving new claims whose truth follows from the truth of our premises.

Since the job of the rules is to allow us to combine and modify well-formed formulas, we have needed a way to state them in terms of placeholders – slots that can be filled with whatever formulas we happen to be working with.

We’ve used variables such as p and q to do that. But, at proof time, we can substitute, for all such variables, any wffs.

The only thing that we must be careful about is that we must substitute uniformly. If one instance of a variable, say p, is replaced by some wff, say $(a \vee b)$, then every instance of p must be replaced by $(a \vee b)$.



https://www.youtube.com/watch?v=gzp6m9a2woI

Distributivity tells us that: $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$

Suppose that we have:

[1] $(A \wedge B) \wedge (C \vee D)$

Then substituting $(A \wedge B)$ for p, C for q, and D for r, we can used Distributivity to prove:

[2] $((A \wedge B) \wedge C) \vee (A \wedge B) \wedge D)$

Big Idea

When we state logical rules, variables are placeholders for arbitrary wffs.

Problems

1. Recall that one form of De Morgan’s Laws is:

   $\neg(p \wedge q) \equiv \neg p \vee \neg q$

   Consider: [1] $\neg((P \wedge Q) \wedge (R \wedge S))$

   a) To apply De Morgan to [1], we should let p equal what?
   b) To apply De Morgan to [1], we should let q equal what?
   c) What is the result of applying De Morgan once to [1]?

Simplification

As we work with logical expressions, we often end up with ones that are longer and messier than they need to be. Before we go farther, it can help a lot to attempt to simplify them. In other words, given an expression *E*, we look for an alternative expression that:

- is logically equivalent to *E*, and
- is simpler in some way, and thus easier to work with, than *E* is.

We have two bags of tools that we can use to do this:

- the Boolean identities that we've just described (plus any others that you want badly enough to prove the correctness of), and
- computation.

So how do we know what tools to use and how to use them? There's no magic answer. Often what we try to do is to transform subexpressions so that we're able to use computation.

In the examples that follow, we'll underline, in each expression, the subexpression that will be changed to create the next line. That should make the process a bit easier to follow.

Suppose that we have: $(p \wedge s) \wedge ((p \vee \neg p) \rightarrow (\neg p \wedge r))$

We can simplify as follows:

[1]	$(p \wedge s) \wedge (\underline{(p \vee \neg p)} \rightarrow (\neg p \wedge r))$	
[2]	$(p \wedge s) \wedge \underline{(T \rightarrow (\neg p \wedge r))}$	Computation
[3]	$(\underline{p \wedge s}) \wedge (\neg p \wedge r)$	Computation

Now we need to get p and $\neg p$ together. So:

[4]	$\underline{(s \wedge p) \wedge (\neg p \wedge r)}$	Commutativity of *and*
[5]	$(\underline{(s \wedge p) \wedge \neg p}) \wedge r$	Associativity of *and*
[6]	$(s \wedge (\underline{p \wedge \neg p})) \wedge r$	Associativity of *and*
[7]	$(\underline{s \wedge F}) \wedge r$	Computation
[8]	$\underline{F \wedge r}$	Computation
[9]	F	Computation

Sometimes, it can take just a few steps to do a really significant simplification.

Suppose that we have: $((p \wedge q) \vee (\neg p \vee \neg q)) \wedge (p \vee r)$.

The trick in this example is to use De Morgan's laws backwards from the way we usually use them. Why? Because, in this case, doing so will create a subexpression of the form $P \vee \neg P$ that can be simplified to T. (More precisely, we'll end up with $(p \wedge q) \vee \neg(p \wedge q)$, but, letting P stand for $(p \wedge q)$, we have $P \vee \neg P$.)

So we can simplify as follows:

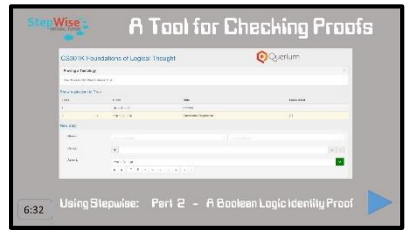
[1]	$((p \wedge q) \vee (\underline{\neg p \vee \neg q})) \wedge (p \vee r)$	
[2]	$(\underline{(p \wedge q) \vee \neg(p \wedge q)}) \wedge (p \vee r)$	De Morgan
[3]	$T \wedge (p \vee r)$	Computation
[4]	$p \vee r$	Computation

When we simplify an expression, we're actually doing a special kind of proof. We're proving that the expression that we started with and the one that we ended up with are, in fact, equivalent. We do that using the identities and computational rules that we've just described.

In the next couple of learning modules, we'll add to our Boolean logic proof arsenal a collection of inference rules. Then we'll see how to exploit combinations of identities, computational rules and inference rules to produce useful proofs.

A Tool for Checking Natural Deduction Proofs

https://www.youtube.com/watch?v=afmo7LK6-bE



https://www.youtube.com/watch?v=fNJ4EQe3RQE

We have built an interactive proof checker that you can use to check your proofs as you are writing them. We can begin using it now, for simplification proofs. Later we'll see that it can also be used for proof that exploit additional inference rules.

The checker needs to be initialized with a particular problem to solve. There isn't a simple interface that lets you create problems and feed them to the checker. But we have created a collection of them that you can work with.

When it's time to do a proof, either as an example in one of our slides, or as part of a problem, you'll see the proof checker show up on your screen.

You can create your proof with very little typing. You can cut an paste from previous lines or from the symbol list at the bottom of the proof area.

To create a proof step, begin by choosing one or two statements from the list of available ones. Initially, there will just be premises. But, as you create new lines in the proof, they too will be available.

Then select a rule from the rule selection tool bar.

Finally enter the line that results from applying the chosen rule to the chosen input(s). Click the green check mark and the checker will test whether your step is valid.

If you click on the funnel (at the left of the rule selection tool bar), the checker will filter the rules and only show you the ones that can be applied to the statement(s) you've selected.

If you have selected a rule, you can click on the wrench (on the right of the rule selection bar) and you'll see what will happen if you apply that rule to the statement(s) you've selected.

Problems

The problems in this section give you a chance to practice using combinations of the identities to simplify Boolean expressions.

1. Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology by using Boolean identities to prove that it is equivalent to T.

2. Simplify: $\neg(r \vee q) \rightarrow \neg(p \wedge (q \wedge s))$ to T.

3. Prove that these two expressions are equivalent:

- $p \vee \neg(q \wedge r)$
- $q \rightarrow (r \rightarrow p)$

Problems

In these problems, we'll explore the relationship between equivalent English sentences and the corresponding equivalent Boolean expressions.

1. Consider the sentence: He was not unaware that she was a student.

Which of the following gives an equivalent sentence and explains the equivalence with one of our identities:

a) He was aware that she was a student. Contrapositive.
b) He was aware that she was a student. Double Negation.
c) He was unaware that she was a student. Double Negation.
d) She was a student. Idempotence.
e) He was aware that she wasn't a student. Conditional Disjunction.

2. Consider the sentence: The Astros and the Phillies can't both win.

Which of the following gives an equivalent sentence and explains the equivalence with one of our identities:

a) The Astros and the Phillies can both win. Contrapositive.
b) The Astros or the Phillies can win. De Morgan.
c) The Astros or the Phillies must lose. De Morgan.
d) The Astros and the Phillies must lose. Contrapositive.
e) The Astros and the Phillies must lose. De Morgan.

3. Consider the sentence: The kitten stays or I'm outta here.

Which of the following gives an equivalent sentence and explains the equivalence with one of our identities:

1. The kitten and I are leaving. De Morgan.
2. If the kitten leaves, I go too. Conditional Disjunction.
3. The kitten and I are both staying. Conditional Disjunction.
4. If the kitten stays, so do I. Commutativity of *or*.
5. If the kitten leaves, I go too. De Morgan.

4. Consider the sentence: Take your umbrella or it will rain.

Which of the following gives an equivalent sentence and explains the equivalence with one of our identities:

a) If you don't take your umbrella, it will rain. Conditional Disjunction.
b) If you take your umbrella, it will rain. Conditional Disjunction.
c) If you take your umbrella, it won't rain. Conditional Disjunction.
d) If it rains, you didn't take your umbrella. Conditional Disjunction.
e) If you don't take your umbrella, it will rain. De Morgan.

Back to Boolean Expressions in Programming

Recall that we've already seen that Boolean expressions play a key role in programming: they let programs respond to different circumstances, different sets of data, etc.

The Boolean identities that we've just proved are true of all Boolean expressions, including the ones in programs. So they can tell us that two Boolean expressions are equivalent and thus that two programs, one using one expression and the other using the other, will also be equivalent (up to possible efficiency issues that may arise if one expression is simpler than another).

Both of these Python programs describe the same, very lenient, way of assigning daily credit to students in a class:

```
if not Late and not Sleeping:
      GetFullCredit
Else:
      Fail

if not (Late or Sleeping):
      GetFullCredit
Else:
      Fail
```

Programmers know that these two programs are equivalent because they know about De Morgan's laws.

Problems

1. Consider the following program:

```
if Hungry and not (Busy or Broke):
      GetFood
```

Consider the following alternative programs:

I.
```
if Hungry and not Busy and not Broke:
      GetFood
```

II.
```
if (Hungry and not Busy) or Broke:
      GetFood
```

III.
```
if not (not Hungry or Busy or Broke):
      GetFood
```

Which of them is/are equivalent to our original program?

Normal Forms

The identities that we've just been working with give us a way to transform a Boolean statement into another, equivalent one. We've seen that we might want to do that, for example, to produce a simpler expression that we'll have an easier time working with.

> Recall that we have just shown that $p \wedge \neg(\neg s \vee p)$ can be simplified to F.

But we can also exploit the identities if we want to assure that all the statements we're working with have some sort of special form. Depending on what we plan to do with the statements, guaranteeing a special form might make life easier.

> For example, we might want to require that all *nots* be atomic, by which we mean that they apply to just a single variable. So, we'd require that:
>
> $\neg(a \vee b \vee c)$
>
> be rewritten (using De Morgan's Laws) as this equivalent statement:
>
> $\neg a \wedge \neg b \wedge \neg c$

The "Atomic Nots" form has the property that every Boolean expression can be rewritten into it.

> But what about requiring that there be no *nots* at all? Now there are things we can't say. For example, there's no way to say:
>
> $a \vee \neg b \vee c$

We'll say that a form constraint is a ***normal form*** for some original set of objects just in case every original object has an equivalent representation that does satisfy the constraint. So:

- We can call "Atomic Nots" a normal form for Boolean expressions.
- We cannot call "No Nots" a normal form for Boolean expressions.

By the way, normal forms are useful in applications that range from logic to parsing computer programs to handling data base queries. The notion of an "equivalent" representation necessarily depends on what our purpose is for working with the objects we're manipulating. For our purposes, it means, "have the same truth value".

Nifty Aside

Conjunctive Normal Form (or CNF) is probably the most widely used normal form for logical expressions. An expression is in CNF if it is the conjunction of disjuncts.

The formula:

$$(p \vee q) \wedge (a \vee q \vee \neg c) \wedge \neg r$$

is in CNF. All *nots* are atomic, all top level operators are *ands*, and inside parentheses there are only *ors*.

CNF is the basis for an important computational logic technique called resolution. It also plays a key role in a large collection of proofs about computational complexity.

Nifty Aside

Disjunctive Normal Form (or DNF) is a sort of opposite of CNF. An expression is in DNF if it is the disjunction of conjuncts.

The formula:

$$(p \wedge q) \vee (a \wedge q \wedge \neg c) \vee \neg r$$

is in DNF. All *nots* are atomic, all top level operators are *ors*, and inside parentheses there are only *ands*.

DNF is the basis for a very useful way to specify database queries.

Boolean Inference Rules

Introduction

In the last section, we looked at identities: ways of transforming a single logical statement into another (presumably more useful) one.

But proofs (in fact, more generally, arguments) require that we reason with multiple statements to see what new conclusions we can draw from an entire set of premises.

Recall the ***Wet Sidewalks*** example. We gave the following names to statements:

R: It's raining.
W: The sidewalks are wet.
S: The sidewalks are slippery.
C: It is important to be careful.

We supplied the following premises:

[1] $R \rightarrow W$ If it's raining then the sidewalks will be wet.
[2] $W \rightarrow S$ If the sidewalks are wet, they will be slippery.
[3] $S \rightarrow C$ If the sidewalks are slippery then it is important to be careful.
[4] R It's raining.

And then we reasoned as follows:

- If it's raining then the sidewalks will be wet. But it is raining. So the sidewalks will be wet.
- If the sidewalks are wet, they will be slippery. But they are wet. So they are slippery.
- If the sidewalks are slippery then it is important to be careful. But they are slippery. So it is important to be careful.

So we then have:

[5] C It is important to be careful.

In this section, we'll formalize the inference (reasoning) rules that will enable us to make arguments (generate proofs) of this sort. We'll start with *modus ponens*, the one we used informally in the ***Wet Sidewalks*** example.

Inference Rules Preserve Truth

As in the case of the identities in the last section, what we're claiming, in our statement of each of these inference rules, is that they are true for all propositions (*p*, *q*, *r*, or even compound expressions containing Boolean operators).

Also, as in the case of the identities, we prove the correctness of each of these rules using truth tables. What we mean by correctness (generally called ***soundness*** in this context) is:

- If a rule is applied to a set of premises *P* and it generates a new statement *q*, then *q* is guaranteed to be true whenever all the elements of *P* are.

Recall that we have other synonyms for this:

- *q* follows from *P*.
- *P* logically implies *q*.
- *P* entails *q*.

Whatever we call it, we must preserve truth. We can describe how to do that by using the same structure that we used when we first introduced the idea of proof. We must show that, if one of our rules allows us to generate *q* from a set of premises *P*, then this is true:

$$(premise_1 \land premise_2 \land premise_3 \land \ldots \land premise_n) \rightarrow q$$

We already know how to prove claims of this sort. Truth tables to the rescue. We will prove the correctness of each of our new inference rules with a truth table.

But first, we'll introduce one more notation that is common when describing inference rules. We'll write:

$$\begin{array}{l} \quad input_1 \\ \quad input_2 \\ \quad \ldots \\ \quad input_n \\ \hline \therefore \; conclusion \end{array}$$

This means that the rule we're defining applies to one or more input statements and allows us to infer the conclusion.

Important note: Each of the expressions that matches a pattern above the inference line must be an entire statement. While it is allowed to apply identities to subexpressions, ***inference rules can apply only to entire statements.***

A List of Inference Rules

Modus Ponens:

$$\begin{array}{ll} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

From premises p and $p \to q$ conclude q.

From $R \to W$ (rain implies wet sidewalks) and R (it's raining), conclude W (wet sidewalks).

We'll prove the soundness of this rule so that we can see how to construct such soundness proofs. What we need to prove is that the following statement is a tautology (it is true for all values of p and q):

$$(p \wedge (p \to q)) \to q$$

Here's the truth table that does that:

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The last column contains all T's. So we have a tautology.

We'll omit the proofs of the rest of the rules presented here. You should prove them yourself. You can use the Truth Table app to do that.

Modus Tollens:

$$\begin{array}{ll} & p \to q \\ & \neg q \\ \hline \therefore & \neg p \end{array} \qquad \begin{array}{ll} & p \to \neg q \\ & q \\ \hline \therefore & \neg p \end{array}$$

The first version says that, from premises $p \to q$ and $\neg q$, conclude $\neg p$. Since p guarantees q, we cannot have p true unless q is true too. Since q is false, p must also be false. We don't actually need the second version, since, in the first one, q can be any logical expression, including a negated one. But the second version may help us avoid an extra application of the Double Negation rule.

From $R \to W$ (rain implies wet sidewalks) and $\neg W$ (the sidewalks aren't wet), conclude $\neg R$ (it's not raining).

Recall the identity that we called Contrapositive: $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$. If we start with:

$(p \rightarrow q)$ and apply the Contrapositive identity, we get:

$(\neg q \rightarrow \neg p)$.

If we're given this, plus $\neg q$, we can use Modus Ponens to get:

$\neg p$.

Modus Tollens lets us do the same thing in a single step.

Sometimes it is helpful to think of reasoning by Modus Tollens as reasoning backward.

Disjunctive Syllogism:

$$\begin{array}{l} p \vee q \\ \neg q \\ \hline \therefore\ p \end{array} \qquad \begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array} \qquad \begin{array}{l} \neg p \vee q \\ p \\ \hline \therefore\ q \end{array} \qquad \begin{array}{l} p \vee \neg q \\ q \\ \hline \therefore\ p \end{array}$$

The first version says that, from premises $p \vee q$ and $\neg p$, conclude p. If at least one of p and q has to be true but we know that q isn't, then p has to be. Again, we don't need any other versions. But they may shorten our proofs. The second version is equivalent since *or* is commutative. The last two, again, may let us do in one step something that would take extra steps involving Double Negation.

$J \vee M$	John or Mary has to drive me to the store.
$\neg J$	"Not I," says John.
M	Mary has to drive.

Here's a real example of Disjunctive Syllogism taken from a news story:

> In commenting on the consistency of Zimmerman's story as well as Zimmerman's apparent relief when falsely told there was a video of the confrontation, Serino said Zimmerman had to be either a pathological liar or telling the truth.
>
> "If we were to take pathological liar off the table…do you think he was telling the truth?" asked defense attorney Mark O'Mara.

Simplification:

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array} \qquad \begin{array}{rl} & p \wedge q \\ \hline \therefore & q \end{array}$$

From the single premise $p \wedge q$, conclude p. Or conclude q. If both p and q are true, then either of them alone must also be true.

$C \wedge K$	Chris and Kate are coming to the party.
C	Chris is coming to the party.

Addition:

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array} \qquad \begin{array}{rl} & p \\ \hline \therefore & q \vee p \end{array}$$

From premise p, conclude that p or q (in either order) must be true for any q. Notice that q can be anything and can be either true or false because, if p is true, q is irrelevant to the truth value of its disjunction with p: $p \vee q$ and $q \vee p$ must also be true.

Conjunction:

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

This seems to be embarrassingly trivial. What is actually happening, however, is that from two separate premises p and q we are allowed to conclude the conjunction of the two of them.

Hypothetical Syllogism:

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

From premises $p \rightarrow q$ and $q \rightarrow r$, conclude $p \rightarrow r$. This looks very similar to a double use of Modus Ponens. It says, given a chain of implications (where the conclusion of one is the premise of the next), the leading premise must imply the final conclusion.

Why is this useful? If we know p then we can use $p \rightarrow q$ to derive q. Then we can use $q \rightarrow r$ to derive r and we're done. But suppose that we don't (yet) know p. This rule lets us conclude that p (if it's true) would imply r. Maybe this tells us, if we're looking for a way to prove r, that we should spend some effort trying to prove p.

Too Early

Suppose that we want to prove that, if I've gotten up before 4:00 am, I'm cranky. (Seems obvious, but proofs are what we're doing here.) Give names to the following statements:

Uby4:	I got up before 4:00am.
Cranky:	I'm cranky.
Tired:	I'm tired.

Assume the following premises:

$Uby4 \rightarrow Tired$	If I got up before 4:00 am, I'm tired.
$Tired \rightarrow Cranky$	If I'm tired, I'm cranky.

Conclude:

$Uby4 \rightarrow Cranky$

Now we're ready to spring into action with a conclusion whenever it happens that I got up ridiculously early.

Problems

1. Give names to the following statements:

C:	Cody is late.
M:	Mary is late.
P:	Peter is late.

Assume the following premises:

[1]	$C \vee M$	Cody or Mary is late.
[2]	$P \vee C$	Peter or Cody is late.
[3]	$\neg M$	Mary isn't late.
[4]	P	Peter is late.

Using our inference rules, we can conclude:

[5]	C	Cody is late.

How did we derive that conclusion?

a) We applied Disjunctive Syllogism to [2] and [4].
b) We applied Modus Tollens to [2] and [4].
c) We applied Modus Ponens to [1] and [4].
d) We applied Disjunctive Syllogism to [1] and [3].
e) We applied Modus Tollens to [1] and [3].

2. Give names to the following statements:

C: Chris likes apples.
M: Mary likes apples.
P: Peter likes apples.

Assume the following premises:

[1]	$C \rightarrow M$	If Chris likes apples, so does Mary.
[2]	$P \rightarrow M$	If Peter likes apples, so does Mary.
[3]	C	Chris likes apples.
[4]	$\neg P$	Peter doesn't like apples.

Using our inference rules, we can conclude:

[5]	M	Mary likes apples.

How did we derive that conclusion?

a) We applied Modus Ponens to [1] and [3].
b) We applied Modus Ponens to [2] and [4].
c) We applied Modus Tollens to [1] and [3].
d) We applied Modus Tollens to [1] and [4].
e) We applied Hypothetical Syllogism to [1] and [2].

3. Give names to the following statements:

C: Chris likes math.
P: Pat likes math.
Y: Taylor likes math.

Assume the following premises:

[1]	$C \rightarrow P$	If Chris likes math, so does Pat.
[2]	$C \rightarrow Y$	If Chris likes math, so does Taylor.
[3]	Y	Taylor likes math.
[4]	$\neg P$	Pat doesn't like math.

Using our inference rules, we can conclude:

[5]	$\neg C$	Chris doesn't like math.

How did we derive that conclusion?

a) We applied Modus Ponens to [1] and [4].
b) We applied Modus Tollens to [1] and [4].
c) We applied Modus Ponens to [2] and [3].
d) We applied Modus Tollens to [2] and [3].
e) We applied Hypothetical Syllogism to [1] and [2].

More Inference Rules

Contradictory Premises:

$$\begin{array}{rl} & p \\ & \neg p \\ \hline \therefore & q \end{array}$$

From premises p and $\neg p$, conclude q for absolutely any q. This may seem the strangest of the lot. The statement q can come from thin air. It may be ridiculous. Nevertheless, for any p, the statement $p \wedge \neg p$ is always false and the definition of *implies* says that a false premise always guarantees a true conclusion (for any conclusion). So we have that, in the face of a contradiction $p \wedge \neg p$, anything may be concluded.

> Recall that we've already proven this. We saw that if we assert:
>
> - The moon is made of green cheese.
> - The moon is not made of green cheese.
>
> then we can prove any of the following:
>
> - Elephants can fly.
> - Elephants cannot fly.
> - The king of France is a unicorn.

Resolution:

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array} \qquad \begin{array}{rl} & p \vee q \\ & r \vee \neg p \\ \hline \therefore & q \vee r \end{array} \qquad \begin{array}{rl} & p \vee q \\ & \neg q \vee r \\ \hline \therefore & p \vee r \end{array} \qquad \begin{array}{rl} & p \vee q \\ & r \vee \neg q \\ \hline \therefore & p \vee r \end{array}$$

The first version says that, from premises $p \vee q$ and $\neg p \vee r$, conclude $q \vee r$. (The other three are equivalent since *or* is commutative.) This is interesting since we never assert any of p, q, or r to be true. But we do know that p must either be true or false. If p is true (and thus $\neg p$ is false) then, to make $\neg p \vee r$ true, r must be true. Alternatively, if p is false (and thus $\neg p$ is true) then, to make $p \vee q$ true, q must be true. Hence either q or r (or both) must be true.

> ***Nifty Aside***
>
> Besides being useful in the sorts of proofs that we're going to do, this rule forms the basis for another proof technique called ***resolution***.

Conditionalization:

$$\begin{array}{ll} & A\text{, a set of premises} \\ & (A \wedge p)\text{ entails } q \\ \hline \therefore & p \rightarrow q \end{array}$$

Suppose that some set A of premises, taken together with one additional premise p, entails q. In other words, q must be true whenever the premises, plus p, are all true. Then, continuing to assume A as our premises, we have that p implies q.

This rule works differently from the others that we have presented. In particular, it describes a derivation process that may require many steps and that may require appeal to any number of other premises.

Note that what we derive here is an implication. We show that, if p is true, then q must also be. We do that in the following steps:

1. Assume that p is true (i.e., add it as a new premise).
2. Reason with it (and with any other required premises or derived statements).
3. Derive q. Note that, at this point, we haven't actually proven q. We've just shown that it must be true if p is.
4. Conclude that (in the context of the rest of our premises) $p \rightarrow q$. When we make explicit the fact that our conclusion rests on the assumption of the extra premise p, we'll say that we've ***discharged*** the premise p.

While Conditionalization doesn't allow us to conclude q, it can be very useful. Once we've got $p \rightarrow q$, there are two things we can do:

- Wait and, if we ever do find out that p is true, we can immediately conclude q.

- Use our new fact as a hint if we're trying to figure out a way to prove q: What we know now is that we should try to prove p.

Slippery

Give names to the following statements:

P: There's precipitation.
Z: It's freezing.
S: It's slippery.

Assume the following premises:

$(P \wedge Z) \rightarrow S$	If there's precipitation and it's freezing, it is slippery.
Z	It's freezing.

Given these two premises, we should be able to prove that, if there's precipitation, it will be slippery. (Pretty reasonable in the winter in many places.) To do this, we introduce the conditional premise:

P There's precipitation.

Now we can assert:

$(P \wedge Z)$

(To get this formally, we use our second premise plus the Conjunction rule.) Using Modus Ponens, along with our first premise, we then have:

S

But we haven't actually proved S. We must discharge the conditional premise P. When we do this, we get:

$P \rightarrow S$ If there's precipitation, it is slippery.

Note that, whenever we use the Conditionalization rule, we must do careful bookkeeping so that we guarantee that all conditional premises have been discharged before we assert a conclusion.

Big Idea

In the appendix, you'll find a Boolean logic "cheat sheet". You may want to keep it handy while working proofs.

Problems

1. Give names to the following statements:

B: We'll watch Bambi.
K: Koko is coming.
P: We'll go on a picnic.
S: We'll watch Shrek.

Assume the following premises:

[1]	$B \vee S$	We'll watch Bambi or Shrek.
[2]	$K \to \neg S \vee \neg P$	If Koko comes, we won't both watch Shrek and go on a picnic.
[3]	K	Koko is coming.

Using the identities and inference rules that we've defined, we can conclude:

[4] $B \vee \neg P$ We'll watch Bambi or we won't go on a picnic.

Which of the following is one way that we could have derived that conclusion:

a) We applied Modus Ponens to [2] and [3], then Hypothetical Syllogism to that result and [1].
b) We applied Resolution to [1] and [2], then Modus Ponens to that result and [3].
c) We applied Resolution to [1] and [2].
d) We applied Modus Ponens to [2] and [3], then resolution to that result and [1].
e) We applied Modus Ponens to [2] and [3], then Disjunctive Syllogism to that result and [1].

A Really Useful Problem Solving Technique – Debugging

> *It is an old maxim of mine that when you have excluded the impossible, whatever remains, however improbable, must be the truth.*
>
> Sherlock Holmes in *The Beryl Coronet*

Consider a very common problem-solving scenario:

Something doesn't work.

Your job is to fix it. Before you can fix it, you have to figure out the cause of the problem.

We do this kind of reasoning every day. It's also a powerful tool for programmers who have to debug their code.

A useful strategy is:

1. Make a list of possible causes.
2. Consider the items on the list one at a time:
    a. If this item appears to be the cause, move on to fix the problem
    b. If this item doesn't appear to be the cause, winnow the list by removing this item.
3. Hope that you don't winnow the list down to empty before finding the cause.

Disjunctive Syllogism is what lets us do this.

Problem: The lamp won't turn on.

Let's give names to the following statements:

Plugged: The lamp is plugged in.
Power: The power is on to the outlet.
Bulb: The bulb is okay.
Broken:The lamp is broken.

Then we might make this claim if we observe that our lamp isn't working and we believe that there are four possible causes of the problem:

$$\neg Plugged \vee (\neg Power \vee (\neg Bulb \vee Broken))$$

(In everyday reasoning, we would skip the parentheses. Later, we will too. But for now, each instance of or must have exactly two arguments. So we'll write it this way and just list the possible causes in the order in which we plan to check them.)

We can now proceed to diagnose the problem. We check that the power is plugged in. It is.

So we reason:

$\neg Plugged \vee (\neg Power \vee (\neg Bulb \vee Broken))$
$Plugged$

$\neg Power \vee (\neg Bulb \vee Broken)$ Disjunctive Syllogism

Next, suppose that we check that the power is on by plugging something else into the outlet. It works. So we reason:

$\neg Power \vee (\neg Bulb \vee Broken)$
$Power$

$\neg Bulb \vee Broken$ Disjunctive Syllogism

And so forth.

Inference Rules Are One Way Streets

Notice an important difference between the identities that we proved in the last section and the inference rules that we've proven here. In the case of the identities, we show that:

$$p \equiv q$$

This claim is symmetric. Substitution can go in either direction. Since *p* and *q* have identical truth values, either can be substituted for the other in any logical expression without changing the expression's truth value.

That's not the case for the inference rules. For them, we've proved only that:

antededents $\rightarrow$ *conclusion*

Generally one side is stronger (true in more circumstances) than the other. So inference just goes in one direction.

Here's a simple example of this. Recall the Simplification rule and the tautology that proves its soundness:

$$\begin{array}{ll} & p \wedge q \\ \hline \therefore & p \end{array}$$

So we can start with:

$C \wedge K$ Chris and Kate are coming to the party.

And conclude:

C Chris is coming to the party.

But we cannot start with C and conclude $C \wedge K$.

Problems

1. Give names to the following statements:

C: There will be cake for dessert.
P: There will be pie for dessert.
CH: There will be chocolate for dessert.

Assume the following premises:

[1]	$C \vee P$	There will be cake or pie for dessert.
[2]	$C \rightarrow CH$	If there's cake it will be chocolate.

We wish to prove:

CH There will be chocolate for dessert.

Here's a proposed proof:

[1]	$C \vee P$	Premise
[2]	$C \rightarrow CH$	Premise
[3]	C	We apply Addition to [1].
[4]	CH	We apply Modus Ponens to [2] and [3].

Which of the following claims is true of this proof? If you think that the proof is wrong, indicate the first place where a mistake is introduced.

a) It is a valid proof.
b) It is not valid. Line 3 is a valid conclusion but the reason given is wrong.
c) It is not valid. Line 3 is not a valid conclusion.
d) It is not valid. Line 4 is a valid conclusion but the reason given is wrong.
e) It is not valid. Line 4 is not a valid conclusion.

Using Inference Rules Correctly

We'll soon see how to approach writing an entire proof that exploits the identities and rules that we've just described. But before we do that, let's just make sure that we understand how individual ones can be applied correctly.

We already know that identities work in both directions, while rules apply only in one.

Here's another important distinction: We can use the identities to make substitutions, both for entire lines in a proof as well as for subexpressions.

> Suppose that we are given:
>
> [1] $(\neg(P \wedge Q)) \rightarrow R$
>
> We can apply De Morgan to the subexpression $\neg(P \wedge Q)$ and thus rewrite [1] as the equivalent sentence:
>
> [2] $(\neg P \vee \neg Q) \rightarrow R$
>
> We can also apply Conditional Disjunction to [1] and thus rewrite it as the equivalent sentence:
>
> [3] $\neg(\neg(P \wedge Q)) \vee R$

The reason that we can apply the identities anywhere is that they are *identities*. They allow us to rewrite one expression as another one with equivalent truth value. So applying them never changes truth values.

On the other hand, the inference rules work only in one direction. They do not rewrite expressions as equivalent ones. We know only that the new one is entailed by the original one. Thus we must be careful to use them exactly as they are written.

> ***Big Idea***
>
> Inference rules may be applied only to entire logical expressions.

Cake and Ice Cream

Give names to the following statements:

C: I get to have cake.
IC: I get to have ice cream.
H: I'll be extra happy.

Suppose that we are given:

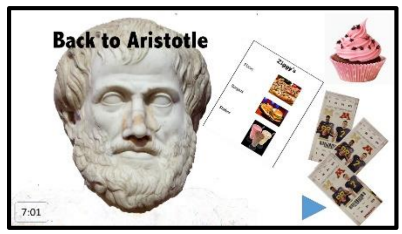
[1] $(C \wedge IC) \rightarrow H$

We may not apply Simplification to $(C \wedge IC)$ to derive C and thus to transform [1] into:

[2] $C \rightarrow H$

Clearly this is a stronger and thus not equivalent claim. We have not been given any reason to believe that I'll be extra happy with only cake.

Now that we've seen both correct and incorrect ways to apply logical rules, let's take another look at some of the examples that we considered at the beginning of this course.



https://www.youtube.com/watch?v=WvFjnuh0F54

Problems

1. Give names to the following premises:

B: Blitzen is pulling the sleigh.
C: Cupid is pulling the sleigh.
D: Dancer is pulling the sleigh.
P: Prancer is pulling the sleigh.
R: Rudolph is pulling the sleigh.

Assume the following premises:

[1] $(P \wedge B) \rightarrow R$
[2] $R \rightarrow (D \vee C)$
[3] $\neg R$
[4] B

Which of the following statements correctly describes a use of one of our inference rules:

a) Use Simplification with [1] to infer $P \rightarrow R$.
b) Use Addition with [2] to infer $(R \vee B) \rightarrow (D \vee C)$.
c) Use Modus Tollens with [1] and [3] to infer $\neg(P \wedge B)$.
d) Use Modus Ponens with [1] and [4] to infer R.
e) Use Contradictory Premises with [3] and [4] to infer C.

Suppose You Want More Rules

By the way, there is nothing “magic” about the particular list of identities and inference rules that we’ve shown here.

If you decide you need to look at some other “Learn Logic” resources (but why would you?), you may find more identities:

- Exportation
- Negation of Conditional

You may also find more inference rules:

- Biconditional Introduction
- Biconditional Elimination
- Disjunction Elimination
- Constructive Dilemma
- Destructive Dilemma
- Absorption
- Law of Clavius

We chose the ones we did because they’re the most useful. And at some point, it just isn’t worth memorizing a longer list.

But if you want more, you can have them. You can create them even in the middle of a proof. Whenever you’re working with Boolean expressions and you’d like to apply a rule that you’re pretty sure is sound but that does not show up on our list, all you have to do is:

1. Write it down. Probably give it a name.

2. Use a truth table (or the natural deduction technique that we’re about to learn) to prove it.

Then it’s yours.

Problems

Indicate, for each of these proposed "identities" and "inference rules", whether it's one we could have added to our list. (Hint: Use a truth table.)

1. Valid identity or not? $((p \wedge q) \rightarrow r) \equiv (p \rightarrow (q \rightarrow r))$

2. Valid rule or not?

$$\begin{array}{ll} & p \rightarrow q \\ & r \rightarrow s \\ & p \vee r \\ \hline \therefore & q \vee s \end{array}$$

3. Valid rule or not?

$$\begin{array}{ll} & p \vee q \\ & (p \wedge q) \rightarrow r \\ \hline \therefore & r \end{array}$$

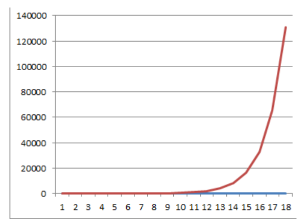
Natural Deduction I

Introduction

If we wanted to make an apple pie, we might start with flour, sugar, apples, and such. This is called cooking "from scratch". Alternatively, we could buy a pre-made pie shell and some apple filling and be 90% done. If we want to prove a Boolean logic theorem, we might build a truth table. On the other hand we might make faster progress using some pre-made theorems: the identities and inference rules of the previous sections.

Use of the identities and inference rules has three benefits over truth tables:

- The size of a truth tables grows as $n \cdot 2^m$, where its corresponding statement has n operators and m variables. For even moderate m and n this can be huge.



- Although a truth table, if completed correctly, produces a bullet-proof argument, there is almost no insight arising from the proof. We may know that something is true but we may have little understanding of *why* that is so.

- The truth table approach does not generalize to more powerful logics, including predicate logic (which we'll explore in the next section). There we'll allow quantified predicates, so we'll be able to say not just that Chris has a mother but also that *all* people have mothers. So, in particular, the truth table approach isn't powerful enough to describe all of what, in everyday life, we call "correct reasoning".

So we actually get a better "proof pie" if we "cheat" and skip the "from scratch" approach.

In much the same way as in the cooking example, we could consider writing computer code in a low level or a high level programing language. The high level language, which lets us build on code that has already been written, makes programming easier and leads to programs that are easier to understand.

We're about to describe a proof technique that we call ***natural deduction***. It will exploit the identities and inference rules that we've just been studying. We call this approach "natural deduction" to emphasize the fact that it corresponds to how humans think.

Problems

1. Consider the expression:

$$(A \wedge P) \vee ((Q \wedge P) \vee (A \wedge B)) \rightarrow R$$

How many rows are there in its truth table?

The Structure of a Natural Deduction Proof

In the last section, we experimented with our reasoning rules and produced what we can call "one-step proofs". Now we want to see how to construct multiple-step proofs that may exploit many premises, as well as several identities and inference rules.

Our proofs will be composed of a sequence of lines. Each line consists of four elements:

[line-number] statement Rule/Premise [referenced line number(s)]

The sequence of lines obeys the following rules:

1. The line numbers start at 1 and proceed sequentially.

2. Each statement is something that we know (at that point in the proof) to be true. So it must be either a premise (including possibly a conditional one) or something that we have just shown to follow from preceding lines of the proof.

3. The third entry in each line justifies the statement on that line. It can simply say, "Premise". Or it can be any of the identities or inference rules we have presented. Or it can be a theorem that we've previously proved.

4. Whenever we use an identity, inference rule or theorem, we must specify the statement(s) to which that identity/rule/theorem applied. So, on any line that doesn't correspond to a premise, we list those earlier line numbers.

Here's a simple example that illustrates the format of a proof that relies on two premises:

[1]	$A \rightarrow \neg T$	Premise	
[2]	A	Premise	
[3]	$\neg T$	Modus Ponens	[1], [2]

Of course, what we want to learn how to do is to construct less trivial proofs (we hope of interesting things).

So, to review: To prove a conclusion C, we must:

1. Choose an appropriate set of premises.
2. Apply our identities and inference rules, starting with one or more of the premises to derive C.

By the way, sometimes when people write proofs, they like to make it clear that they've really done it. The conclusion has been proved. So they may add one last line to their proof:

Q. E. D.

Q.E.D. is an abbreviation for the Latin expression *quod erat demonstrandum*, which means, "that which was to be proved." You can use it if you like.

English Aside

Q.E.D. is now sometimes used in regular English to mean something like, "And so there. End of discussion. I've proved it. I'm right." And it's been translated into a lot of other languages.

Problems

1. Consider the following simple proof:

[1]	$A \rightarrow B$	Premise	
[2]	$K \rightarrow \neg B$	Premise	
[3]	K	Premise	
[4]	____________________	Modus Ponens	[2], [3]
[5]	$\neg A$	____________________	[1], [4]

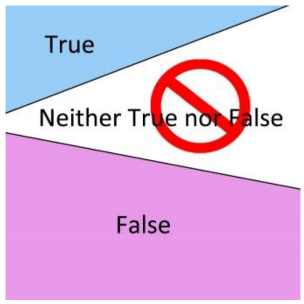
a) What expression should fill in the blank in line 4?

b) What rule should fill in the blank in line 5?

Law of the Excluded Middle

Most of the premises that you'll use in your proofs will be ones that you've chosen to correspond to things that are true in the problem domain you're working in. There is one premise, though, that we'll mention here because it's very general and is not infrequently useful. It is:

Law of the Excluded Middle: $p \vee \neg p$.



In other words, for any statement p, either it's true or false. Either it or its negation must be true. There's nothing "in the middle".

Notice, by the way, that $p \vee \neg p$ is a tautology. It's always true. (That's why it's always a safe premise.) We've chosen to exhibit it here and give it a name because it actually comes up in arguments. We've used it a few times already in this course. But, in fact, it's always safe to use as a premise any tautology. Such a premise will never make it possible to prove something that wasn't provable (perhaps less directly) some other way.

> ***Nifty Aside:***
>
> The Law of the Excluded Middle is noncontroversial in Boolean logic. It's a tautology. But we should point out that, when we move on to predicate (quantified) logic, there are approaches that do not take it as an axiom. We will. But you should know that not everyone does.

English Aside

When we're talking in English, rather than writing in logic, it is frequently less obvious that the Law of the Excluded Middle applies. There are many reasons for this including:

- Many concepts are fuzzy. It's generally quite clear what a cat is. So if we let *C* stand for the claim that Lucy is a cat, we probably won't get in trouble when we try to assign it a truth value (*T* or *F*). But now suppose that Brett is 11 months old. Is it true or false that Brett is a baby? Maybe baby is right. Or maybe a toddler, but not a baby. A person might say, "Sort of".

- Recall that a presupposition is something that a speaker assumes to be true but doesn't explicitly mention. When presuppositions are false, it's hard to assign any truth value. Should we assign T or F to, "The king of France has red hair"? (Exactly who is "the king of France?")

- "Not" can do many different things in English. Consider the sentence, "Travis likes Jody". To negate that in an unambiguous way, we'd have to say, "It's not the case that Travis likes Jody." If we do that, then one or the other of the sentences is true (although even here we could be in trouble if Travis admits to having a lot of conflicting feelings about Jody.) But sentences like our second one are rare. We'd be much more likely to say, "Travis doesn't like Jody", which is likely to be interpreted as, "Travis dislikes Jody". But now it could easily happen that neither "Travis likes Jody," nor "Travis dislikes Jody," is true. It's possible that Travis doesn't know Jody and has no opinion about her. Yet both of these sentences assert that Travis does have some opinion. So they could both be false.

We'll have more to say about these issues later, when we talk about the complex ways in which English sentences map to logical ones. For now, however, we'll take the Law of the Excluded middle as a premise.

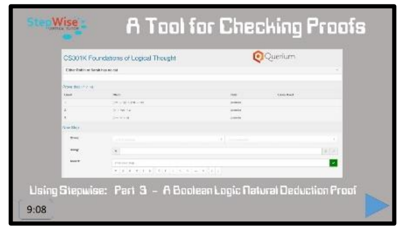
Creating Natural Deduction Proofs

We're going to walk through the process of constructing natural deduction proofs for a collection of representative examples.

For each of these problems, we suggest that you first try to do the proof yourself. You can do this with StepWise, our interactive proof checker.

You can also watch a video in which we walk through the construction of a proof.

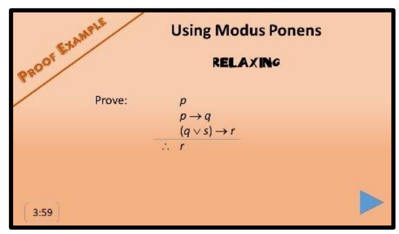
https://www.youtube.com/watch?v=PUVuBoLlsY4



Forward Reasoning – Modus Ponens Proof Example: Relaxing

Relaxing:

We'll start with a simple example. We'll do this one with just symbols (*p*, *q*, *r*, and *s*) so that we're not distracted by a particular real world problem. At the end, we'll suggest such a problem that could correspond to this generic proof.



https://www.youtube.com/watch?v=gj5B4w6LZM0

Prove:

$$\begin{array}{ll} & p \\ & p \to q \\ & (q \vee s) \to r \\ \hline \therefore & r \end{array}$$

You should try to do this proof yourself:

You can also watch our video, which will outline a strategy for creating a proof.

On the next page, you'll find a summary of the approach that is described in the video.

We are given three hypotheses and, although we do not know if we will use them all, let's include them all in the proof – if we notice some are superfluous, we can delete them later. So we can write the first three lines of our proof of r:

[1]	p	Premise	
[2]	$p \to q$	Premise	
[3]	$(q \vee s) \to r$	Premise	

Now let's do some strategizing. The first thing to notice is that (on the basis of what our premises tell us) the only way to conclude r is first to derive $q \vee s$. The role of s is perplexing – it appears nowhere else. It's one of those "out of thin air" sorts of statements. Two of our inference rules allow for statements to be introduced out of thin air: Addition and Contradictory Premises. Using Contradictory Premises is fairly rare since it requires a contradiction – and a glance at the premises doesn't suggest any contradiction. Thus it appears that the use of Addition is going to be one key to this proof.

We need to obtain q somehow before we can use addition to get $q \vee s$. But that's easy from the premises, by using Modus Ponens. Thus the next line of the proof is:

[4]	q	Modus Ponens	[1], [2]

Remember that [1], [2] indicates that we have used lines 1 and 2.
Now it is easy to get $q \vee s$ using Addition:

[5]	$q \vee s$	Addition	[4]

Finally, another use of Modus Ponens gets our conclusion:

[6]	r	Modus Ponens	[3], [5]

Notice, by the way, that we did use all of the hypotheses. We could check by seeing if each of the premise line numbers [1], [2], and [3] in this case, appears someplace in the rightmost column. Here's our complete proof:

[1]	p	Premise	
[2]	$p \to q$	Premise	
[3]	$(q \vee s) \to r$	Premise	
[4]	q	Modus Ponens	[1], [2]
[5]	$q \vee s$	Addition	[4]
[6]	r	Modus Ponens	[3], [5]

By the way, this example could have come from a more real world seeming problem: Suppose we are given that **P**eter is home, the home will be **Q**uiet when **P**eter is home, and if the home is **Q**uiet or it is Tue**S**day it's easy to **R**elax. We prove that it's easy to **R**elax.

Disjunctive Syllogism Proof Example: Who Drives Me?

Who Drives Me?

Next, let's use natural deduction to prove an extension of one of the claims that we've already proved with a truth table. We'll pick up where we left off with the <ex Who Drives Me> example. We'll assign the following names to basic statements:

J: John must drive me to the store.
M: Mary must drive me to the store.
L: John will be late for work.
G: Mary must buy gas.
Y: Mary must have money.
W: Mary must work at a paying job.

(Note that we've simplified reality to assert that Mary must have a paying job in order to have money.)

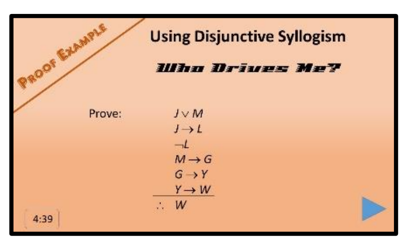
Our premises will be:

[1] $J \vee M$ John or Mary must drive me to the store.
[2] $J \rightarrow L$ If John drives me to the store, he will be late for work.
[3] $\neg L$ John cannot be late for work.
[4] $M \rightarrow G$ If Mary must drive me to the store, she must buy gas.
[5] $G \rightarrow Y$ If Mary must buy gas, she must have money.
[6] $Y \rightarrow W$ If Mary must have money, then she must work at a paying job.

We'd like to prove:

W Mary must work at a paying job.

Again, you should try to do this proof yourself.



You can also watch our video, which will outline a strategy for creating a proof.

https://www.youtube.com/watch?v=vr2rndev_Bo

On the next page, you'll find a summary of the approach that is described in the video.

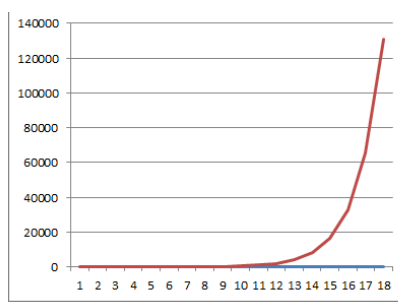
Disjunctive Syllogism Proof Example: Who Drives Me? - Continued

How shall we start? We notice that the only one of our premises that mentions W is [6]. In order to use it (with Modus Ponens), we'll have to know Y. We could derive Y using [5] if we knew G. We could derive G from [4] if we knew M. How can we prove M? The only premise that mentions M is [1], so somehow we're going to have to use it. In order to use it to produce M by itself, we could use Disjunctive Syllogism. But to do that, we'd have to know $\neg J$. Can we prove that? Sure. We can apply Modus Tollens to [2] and [3]. Okay, so we've got a plan. To come up with it, we reasoned backward from what we wanted to prove.

But this is important: While we walked backward to figure out what to do, the actual proof must proceed forward from the premises to the conclusions. Let's do the first step:

[1]	$J \vee M$	Premise	
[2]	$J \rightarrow L$	Premise	
[3]	$\neg L$	Premise	
[4]	$M \rightarrow G$	Premise	
[5]	$G \rightarrow Y$	Premise	
[6]	$Y \rightarrow W$	Premise	
[7]	$\neg J$	Modus Tollens	[2], [3]
[8]	M	Disjunctive Syllogism	[7], [1]
[9]	G	Modus Ponens	[4], [8]
[10]	Y	Modus Ponens	[5], [9]
[11]	W	Modus Ponens	[6], [10]

Notice that this example is a poster child for natural deduction as an alternative to proof by truth table, at least when the proofs have to be constructed by people. It uses six variables. So the truth table that we'd have to build for it would have $2^6 = 64$ rows and 13 columns, for a total of 832 entries. While automatic proving systems can easily handle truth tables that are orders of magnitude bigger than that, we sure don't want to have to write them out by hand.



Conditional Disjunction Proof Problem

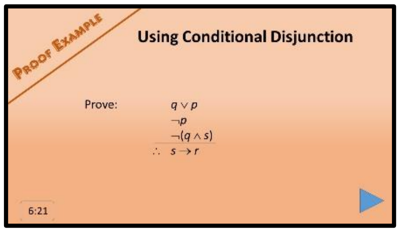
Prove: $q \vee p$
$\neg p$
$\neg(q \wedge s)$

$\therefore \quad s \rightarrow r$

(Hint: There are two reasonable and quite different ways to do this proof. One involves Conditionalization. We'll say more about that soon. So you probably want to do this one without Conditionalization. Instead, you may want to use both Conditional Disjunction and Addition. Recall that Addition lets you introduce a variable "out of thin air". Look at the statement of this problem to see why you might need to do that.)

You should do this proof yourself.

You can also watch our video, which will outline our strategy for doing this.



https://www.youtube.com/watch?v=-4ywUW61wJk

Problems

1. Prove: $R \rightarrow W$
$W \rightarrow S$
$S \rightarrow C$
R
$C \rightarrow I$

$\therefore \quad I$

(Hint: This one is easy. Focus on how you can use Modus Ponens.)

2. Prove: $R \rightarrow W$
$W \rightarrow S$
$S \rightarrow C$
R
$W \rightarrow Y$

$\therefore \quad R \wedge Y$

(Hint: What rule lets you create a conjunction out of two or more statements you already have?)

Backward Reasoning – Modus Tollens Proof Problem: Election

Election:

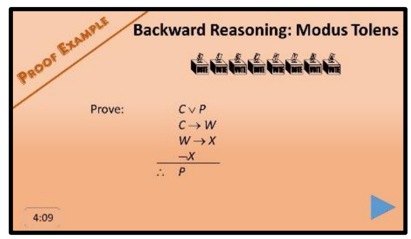
Assign the following names to basic statements:

C: I endorse Carol.
P: I endorse Peter.
W: Carol will win.
X: Taxes are cut.

Prove:	$C \vee P$	I must endorse Carol or Peter.
	$C \rightarrow W$	If I endorse Carol, she will win.
	$W \rightarrow X$	If she wins, taxes will be cut.
	$\neg X$	Taxes cannot be cut.
$\therefore$	P	I must endorse Peter.

You should do this proof yourself.

You can also watch our video, which will outline our strategy for doing this.



https://www.youtube.com/watch?v=irTEOkEN8qI

Problems

1. Prove:

$$\begin{array}{ll} & H \rightarrow (S \vee G) \\ & \neg G \\ & \underline{H \qquad\qquad} \\ \therefore & S \end{array}$$

2. Prove:

$$\begin{array}{ll} & \neg p \vee r \\ & r \rightarrow s \\ & \underline{\neg s \qquad\qquad} \\ \therefore & \neg p \end{array}$$

(Hint: For this one, you're not going to be able to reason forward with Modus Ponens. You'll want to reason "backwards". What rule lets you do that?)

3. Prove:

$$\begin{array}{ll} & \underline{p \vee \neg(q \wedge r)} \\ \therefore & q \rightarrow (r \rightarrow p) \end{array}$$

(Hint: You can use Conditional Disjunction to turn *or*s into implications.)

4. Let's return yet again to a famous ***Catch-22*** situation. We've given names to the following statements:

C: I'm crazy.
R: I've requested a mental health discharge from the Army.
E: I'm eligible for a mental health discharge from the Army.

In Joseph Heller's book, the Army has two rules about this. We have encoded them as premises as follows:

[1] $E \rightarrow C \wedge R$ Only way to be eligible is to be crazy and request the discharge.
[2] $R \rightarrow \neg C$ I'm not crazy if I've requested the discharge.

We want to prove that it's not possible that I'm eligible for a discharge. (And, since we could do this same proof for anyone else, there can never be any of these discharges.)

So prove:

$$\begin{array}{l} E \rightarrow C \wedge R \\ \underline{R \rightarrow \neg C \qquad} \\ \therefore \neg E \end{array}$$

Natural Deduction II

Conditionalization

We are about to do a couple of examples that exploit the Conditionalization rule. Before we do that, let's review it. Recall:

https://www.youtube.com/watch?v=aR7IHkD85L8

Conditionalization:

$$\begin{array}{ll} & A\text{, a set of premises} \\ & (A \wedge p) \text{ entails } q \\ \hline \therefore & p \to q \end{array}$$

.

Conditional Premise Proof Problem: Party

Party:

If we come home early and then, if we go to sleep early, we will get up early. Either we will come home early or we will go to the party. In either case, we will go to sleep early. Therefore, if we do not go to the party, we will get up early.

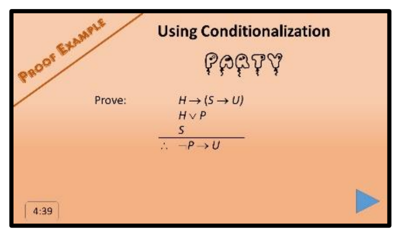
Assign the following names to basic statements:

H: We come **H**ome early.
S: We go to **Sl**eep early.
U: We will get **U**p early.
P: We will go to the **P**arty.

Prove :	$H \to (S \to U)$	If we come home early, and then if we go to sleep early, we will get up early.
	$H \vee P$	Either we will come home early or we will go to the party.
	S	We will go to sleep early.
$\therefore$	$\neg P \to U$	If we do not go to the party, we will get up early.

You should do this proof yourself.

You can also watch our video, which will outline a strategy for creating a proof.



https://www.youtube.com/watch?v=77q6XjPlZRw

Conditional Premise Proof Problem: Stay in Austin

Stay in Austin

If Kelly gets her degree, she will get a good job and a good house. If she has a good job she will live in either Palo Alto or Austin. Kelly will not live in Palo Alto. Therefore, if she gets a degree she will live in Austin.

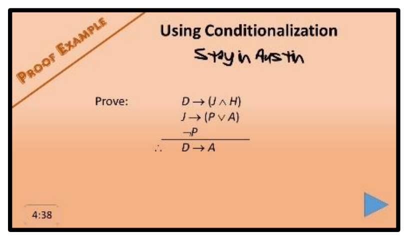
Assign the following names to basic statements:

D:	Kelly gets her **D**egree.
J:	Kelly will get a good **J**ob.
H:	Kelly will get a good **H**ouse.
P:	Kelly will live in **P**alo Alto.
A:	Kelly will live in **A**ustin.

Prove:	$D \to (J \wedge H)$	If Kelly gets degree, she will get a good job and house.
	$J \to (P \vee A)$	If Kelly has a good job she will live in Palo Alto or Austin.
	$\neg P$	Kelly will not live in Palo Alto.
$\therefore$	$D \to A$	If Kelly gets a degree she will live in Austin.

You should try this problem yourself.

You can also watch our video, which will outline a strategy for creating a proof.



https://www.youtube.com/watch?v=EtACAymnx4w

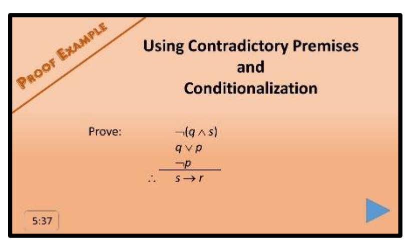
Contradictory Premises and Conditionalization Proof Problem

Prove:

$$\begin{array}{ll} & \neg(q \wedge s) \\ & q \vee p \\ & \neg p \\ \hline \therefore & s \to r \end{array}$$

Notice that you've seen this problem before. This time, use Conditionalization to complete your proof.

You can also watch our video, which will outline our strategy for doing this.



https://www.youtube.com/watch?v=UVxnodlKAn0

Conditionalization Problems

1. Prove: $(A \wedge (\neg B \wedge \neg C)) \rightarrow (A \vee \neg(B \wedge C))$

(Hint: Think about using Simplification and/or Addition to remove terms from a conjunction or add terms to a disjunction.)

2. We know that the show starts at 7 or 8 on some day. We want to show that if it is not true that the show starts on Saturday at 7 then, if it starts at Saturday at a time other than 8, they will burn down the theater.

Assign the following names to basic statements:

E: Show is ***E***arly (at 7).
L: Show is ***L***ate (at 8).
S: Show is on ***S***aturday.
B: Theater ***B***urns down.

Prove:

$$\begin{array}{ll} & E \vee L \\ \hline \therefore & (\neg(E \wedge S)) \rightarrow ((S \wedge \neg L) \rightarrow B) \end{array}$$

(Hint: Employ the Conditionalization rule more than once.)

3. Either Joe or Mary or Sally will go to New York. If Paul stays home, then Joe will not go. Therefore, if Paul stays home and Mary does not go to New York, Sally must go to New York.

Assign the following names to basic statements:

J: Joe will go to New York.
M: Mary will go to New York.
S: Sally will go to New York.
P: Paul stays home.

Prove:

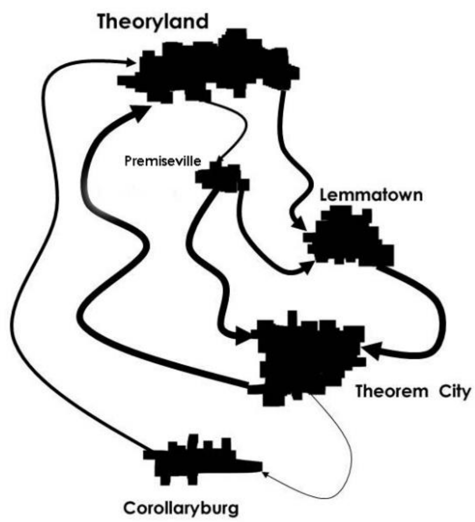
$J \vee (M \vee S)$	Joe or Mary or Sally will go to New York.
$P \rightarrow \neg J$	If Paul stays home, Joe will not go to New York.
$\therefore (P \wedge \neg M) \rightarrow S$	If Paul stays home and Mary doesn't go to NY, Sally must go to New York.

4. Prove that the following claim is a tautology (in other words, derive it without any premises):

$$(\neg p \wedge (\neg q \rightarrow p)) \rightarrow q$$

Theorem upon Theorem (Again): Using Lemmas and Corollaries

Recall that back in our discussion of truth tables, we introduced the idea that, when confronted with a nontrivial problem, we could prove some intermediate results and then use those results (theorems) as though they were additional premises.



This same idea is important when we're building natural deduction proofs of interesting claims. In fact, there are two notions of "related theorems" that we'll find useful. They are typically given names:

- A ***lemma*** is a theorem that we prove along the way toward a proof of a bigger claim. Once we've proved a lemma, we can, of course use it in other proofs as well.

- A ***corollary*** is a theorem that's a not too distant spinoff of a theorem we've just proved. Typically the proof of a corollary is fairly straightforward. But the corollary may put the claim of the theorem in a more useful form.

Problems

1. Suppose that we have the following premises:

[1] $a \rightarrow p$
[2] $w \rightarrow r$
[3] $s \rightarrow x$
[4] $s \rightarrow w$
[5] $(x \wedge w) \rightarrow a$
[6] $(x \wedge w) \rightarrow b$
[7] $(x \wedge w) \rightarrow c$

Further suppose that *a*, *b*, and *c* are all important conclusions that we'd very much like to be able to prove. Which one of the following possible lemmas would it make sense to try to prove:

a) a
b) r
c) s
d) x
e) w

2. Suppose that we have the following premises:

[1] $\neg e$
[2] $p \rightarrow \neg q$
[3] $p \rightarrow v$
[4] $q \rightarrow s$
[5] $q \rightarrow b$
[6] $\neg r \vee w$
[7] $r \rightarrow c$
[8] $s \rightarrow (a \vee e)$
[9] $s \rightarrow d$
[10] $s \rightarrow r$

Further suppose that *a*, *b*, *c*, and *d* are all important conclusions that we'd very much like to be able to prove. Which one of the following possible lemmas would it make sense to try to prove:

a) p
b) q
c) r
d) s
e) w

Soundness and Completeness

Introduction

The proofs that we've done so far are syntactic objects. We write down a set of premises. Then we do symbol manipulation. We've seen two ways to do that:

We can construct a natural proof by applying the identities and inference rules that we've described. Then out comes a final line that we term a "theorem".

[1]	p	Premise	
[2]	$p \rightarrow q$	Premise	
[3]	$(q \vee s) \rightarrow r$	Premise	
[4]	q	Modus Ponens	[1], [2]
[5]	$q \vee s$	Addition	[4]
[6]	r	Modus Ponens	[3], [5]

R	W	R → W	(R ∧ (R → W)	(R ∧ (R → W)) → W
T	T	T	T	T
T	F	F	F	T
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T
F	T	T	F	T
F	F	T	F	T

Alternatively, we can build a big truth table using the small truth tables that serve as the definitions of the logical operators. For convenience in the following discussion, let's consider "proof by truth table" to be just one more inference rule (even though it feels quite different from the "natural" ones).

Getting at Truth – An Inference System that is Sound and Complete

The big deal though is that we want proofs to tell us something about *truth*. They can do that if we design our inference rules appropriately. To see how to do that, we need two definitions:

- We'll say that an inference rule is ***sound*** if and only if, whenever it is applied to a set *P* of premises, any conclusion that it produces is entailed by *P* (i.e., it must be true whenever *P* is).

- A set of inference rules *R* is ***complete*** if and only if, given any set *P* of premises, all statements that are entailed by *P* can be proved by applying the rules in *R*.

If we can define a set of inference rules that is both sound and complete then the set of theorems that can be proved from *P* will exactly correspond to the set of statements that must be true whenever all the premises in *P* are true.

So do we have a set of sound and complete inference rules for Boolean logic? The answer is yes. Proof by truth table (all by itself) is both sound and complete (even if not convenient). Moving on to natural deduction: All our rules are sound. We know this because we've shown the truth tables that prove them to be so. (Thus it's never possible to use one of them to derive a conclusion that doesn't follow from the premises.) And, taken together, they are complete. (So it's always possible, given a statement *S* that does in fact follow from our premises, to construct a proof of *S* using our rules.)

Big Idea

The Boolean logic inference engine that we have just studied is both sound and complete.

Completeness is important if we don't want to be stuck, staring at something we know to be true, but unable to construct a proof.

Soundness is even more fundamental. If we admitted even a single unsound inference rule, proof would no longer tell us anything about truth (our ultimate goal in this whole endeavor). To see why that's so, suppose that we added, say, this new rule:

Flimflam:

$$\begin{array}{rc} & p \\ \hline \therefore & p \wedge q \end{array}$$

Thus, from premise *p* conclude *p and* anything we want.

Recall that the Addition rule tells us that, from *p*, we can infer $p \vee q$ for any statement *q*. This new rule, Flimflam, lets us do a similar thing except that we can introduce *and* instead of just *or*.

Let's see what we can do with it:

[1]	Q	Premise	
[2]	$Q \wedge \neg Q$	Flimflam	[1]*
[3]	F	Computation	[2]

* Note that Flimflam lets us conjoin Q with absolutely any statement. We happen to have picked the statement $\neg Q$ in order to get a very simple example of the damage that Flimflam has wrought.

So we see: Flimflam lets us derive False.

And we can keep going:

[4]	The moon is made of green cheese.	Contradictory Premises	[2]

This is precisely the kind of flawed reasoning that sound rules will not let us do.

That we have a set of sound and complete inference rules is great news. Given any problem that we can represent in Boolean logic, we can, in principle, prove all and only the conclusions that follow from the premises that we choose.

Problems

1. Given a set of premises *P* and a conclusion *C*: if we can use the natural deduction system that we've just described to conclude *C* from *P*, is it possible for all the premises in *P* to be true but *C* to be false?

a) Yes.
b) No.

2. Given a set of premises *P* and a conclusion *C* that is entailed by *P* (i.e., must be true whenever *P* is), is it possible that, in our system, there is no proof of *C*?

a) Yes.
b) No.

3. Assume that a set of premises *P* entails a conclusion *C*. Suppose that we add a new premise *p* to *P*. Is it possible that, in our system, there is no proof of *C*?

a) Yes.
b) No.

4. Suppose that a set of premises *P* does not entail a conclusion *C*. Is it possible that our reasoning system could produce a proof of *C* from *P*?

a) Yes.
b) No.

5. Suppose that *P* is a set of premises and *p* is one more. If ($p \wedge P$) entails some conclusion *C*, is it possible that our reasoning system could be unable to produce a proof of *C* given just the premises in *P*?

a) Yes
b) No

Getting at Truth – Sound and Valid Arguments

So now we know that we've got a set of inference rules that, given some set of premises *P*, let us prove:

- all, and
- only

the statements that follow from *P*.

We'll say that an argument (proof) is ***valid*** provided that every one of its steps can be justified by a sound inference rule. Sometimes, when presented with such an argument, we'll say, "Its reasoning is valid."

But does this mean that the conclusion of a valid argument is necessarily *true*? Unfortunately, no.

Give names to the following statements:

L: Lucy is a unicorn.
H: Lucy has a large horn on her head.

Suppose that we have the following premises:

[1]	L	Lucy is a unicorn
[2]	$L \rightarrow H$	If Lucy is a unicorn then she has a large horn on her head.

Then, using Modus Ponens, we have:

[3]	H	Lucy has a large horn on her head.

But Lucy has no horn. The problem is that she's a cat, not a unicorn. Our reasoning is valid. But we've proven something that isn't true.

We'll say that an argument (proof) is ***sound*** provided that it is valid *and* that its premises are true (in whatever world we are reasoning about). The Lucy argument that we just gave is valid but not sound.

To get at truth, we must construct sound arguments. They must start from premises that are true and their reasoning must be valid.

By the way, in case you're a bit confused about terminology here, you're not alone. There is an unfortunate (but so conventional that we cannot ignore it) use of the word "sound" to mean one thing when applied to inference rules and another thing when applied to entire arguments:

- An inference rule is sound just in case it *preserves* truth. In other words, it can derive only conclusions that follow from the premises.
- An argument (proof) is sound just in case truth is both *introduced* by the premises and *preserved* by the argument.

Sorry about that. But don't worry. The key thing is that good arguments possess both properties. We are focusing on developing sound inference rules. When people go to apply those rules to help them reason about real problems, it's up to them to choose premises that make sense in their problem domains.

Problems

1. Assume the following premises:

[1]	C	Lucy is a cat.
[2]	M	Lucy lives on Mars.
[3]	$M \rightarrow N$	If Lucy lives on Mars, there is catnip on Mars.
[4]	$C \rightarrow P$	If Lucy is a cat, Lucy purrs.
[5]	$G \rightarrow P$	If Lucy is a tiger, Lucy purrs.

(Part 1) Consider the following conclusion that we would like to prove:

[6] N There is catnip on Mars.

Which of the following statements is true:

a) It is not possible to construct a valid argument to support this conclusion.
b) It is possible to construct a valid argument, but not a sound one, to support this conclusion.
c) It is possible to construct a sound argument to support this conclusion.

(Part 2) Consider the following conclusion that we would like to prove:

[7] P Lucy purrs.

Which of the following statements is true:

a) It is not possible to construct a valid argument to support this conclusion.
b) It is possible to construct a valid argument, but not a sound one, to support this conclusion.
c) It is possible to construct a sound argument to support this conclusion.

(Part 3) Consider the following conclusion that we would like to prove:

[7] G Lucy is a tiger.

Which of the following statements is true:

a) It is not possible to construct a valid argument to support this conclusion
b) It is possible to construct a valid argument, but not a sound one, to support this conclusion.
c) It is possible to construct a sound argument to support this conclusion.

Appendices

Boolean Identities

Double Negation:	p	$\equiv$	$\neg(\neg p)$
Equivalence:	$(p \equiv q)$	$\equiv$	$(p \to q) \wedge (q \to p)$
Idempotence:	$(p \wedge p)$	$\equiv$	p
	$(p \vee p)$	$\equiv$	p
DeMorgan$_1$:	$(\neg(p \wedge q))$	$\equiv$	$(\neg p \vee \neg q)$
DeMorgan$_2$:	$\neg(p \vee q)$	$\equiv$	$(\neg p \wedge \neg q)$
Commutativity of *or*:	$(p \vee q)$	$\equiv$	$(q \vee p)$
Commutativity of *and*:	$(p \wedge q)$	$\equiv$	$(q \wedge p)$
Associativity of *or*:	$(p \vee (q \vee r))$	$\equiv$	$((p \vee q) \vee r)$
Associativity of *and*:	$(p \wedge (q \wedge r))$	$\equiv$	$((p \wedge q) \wedge r)$
Distributivity of *and* over *or*:	$(p \wedge (q \vee r))$	$\equiv$	$((p \wedge q) \vee (p \wedge r))$
Distributivity of *or* over *and*:	$(p \vee (q \wedge r))$	$\equiv$	$((p \vee q) \wedge (p \vee r))$
Conditional Disjunction:	$(p \to q)$	$\equiv$	$(\neg p \vee q)$
Contrapositive:	$(p \to q)$	$\equiv$	$(\neg q \to \neg p)$

Boolean Inference Rules

Modus Ponens:	From p and $p \to q$,	infer q	
Modus Tollens:	From $p \to q$ and $\neg q$,	infer $\neg p$	. . .
Disjunctive Syllogism:	From $p \vee q$ and $\neg q$,	infer p	. . .
Simplification:	From $p \wedge q$,	infer p	. . .
Addition:	From p,	infer $p \vee q$	. . .
Conjunction:	From p and q,	infer $p \wedge q$	
Hypothetical Syllogism:	From $p \to q$ and $q \to r$,	infer $p \to r$	
Contradictory Premises:	From p and $\neg p$,	infer q	
Resolution:	From $p \vee q$ and $\neg p \vee r$,	infer $q \vee r$	. . .
Conditionalization:	Assume premises A. Then, if $(A \wedge p)$ entails q,	infer $p \to q$	

Computation

$p \vee \neg p$	$\equiv$	T	$\neg p \vee p$	$\equiv$	T
$p \wedge \neg p$	$\equiv$	F	$\neg p \wedge p$	$\equiv$	F
$p \vee T$	$\equiv$	T	$T \vee p$	$\equiv$	T
$p \vee F$	$\equiv$	p	$F \vee p$	$\equiv$	p
$p \wedge T$	$\equiv$	p	$T \wedge p$	$\equiv$	p
$p \wedge F$	$\equiv$	F	$F \wedge p$	$\equiv$	F
$p \vee \neg p$	$\equiv$	T	$\neg p \vee p$	$\equiv$	T
$p \wedge \neg p$	$\equiv$	F	$\neg p \wedge p$	$\equiv$	F
$p \vee T$	$\equiv$	T	$T \vee p$	$\equiv$	T
$p \vee F$	$\equiv$	p	$F \vee p$	$\equiv$	p
$p \wedge T$	$\equiv$	p	$T \wedge p$	$\equiv$	p
$p \wedge F$	$\equiv$	F	$F \wedge p$	$\equiv$	F

A Useful Axiom

Law of the Excluded Middle: $p \vee \neg p$