Section 2.4 - Properties of Matrix-Matrix Multiplication

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Matrix-Matrix Multiplication is Associative

Let A, B, and C be matrices of conforming dimensions. Then

$$(AB)C = A(BC).$$

Proof

Let e_j equal the jth unit basis vector. Then

$$(AB)Ce_j = (AB)c_j = A(Bc_j) = A(BCe_j) = A(BC)e_j.$$

Thus, the columns of $(AB)C$ equal the columns of $A(BC)$, making the two matrices equal.

Matrix-Matrix Multiplication is Distributive

Let A, B, and C be matrices of conforming dimensions. Then

$$(A + B)C = AC + BC \quad \text{and} \quad A(B + C) = AB + AC.$$

Note: Matrix-matrix multiplication does not commute

Only in very rare cases does AB equal BA. Indeed, the matrix dimensions may not even be conformal.

Theorem

Let $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$. Then

$$(AB)^T = B^T A^T.$$

Lemma

Let $A \in \mathbb{R}^{m \times n}$. Then $e_i^T A^T = (Ae_i)^T$ and $A^T e_j = (e_j^T A)^T$.

Proof of Lemma

The proof of this lemma is pretty obvious: The ith row of A^T is clearly the ith column of A, but viewed as a row, etc.

Proof of Theorem that $(AB)^T = B^T A^T$

We prove that the (i,j) element of $(AB)^T$ equals the (i,j) element of $(B^T A^T)$:

$$e_i^T (AB)^T e_j$$

$$= < (i,j) \text{ element of } C \text{ equals } (j,i) \text{ element of } C^T >$$

$$e_j^T (AB) e_i$$

$$= < \text{Associativity of matrix multiplication} >$$

$$(e_j^T A)(B e_i)$$

$$= < x^T y = y^T x >$$

$$(B e_i)^T (e_j^T A)^T$$

$$= < \text{Lemma} >$$

$$e_i^T (B^T A^T) e_j.$$