Problem

Consider the differential equations

$$\frac{dx_1}{dt} = -x_2 + a_0x_1^3 + a_1x_1^2x_2 + a_2x_1x_2^2 + a_3x_2^3$$

$$\frac{dx_2}{dt} = x_1 + b_0x_2^3 + b_1x_2^2x_1 + b_2x_2x_1^2 + b_3x_1^3$$

Show that the solution $x_1 = x_2 = 0$ is asymptotically stable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $\| \underline{x}(0) \| < \delta$ we have $\lim_{t \to \infty} \underline{x}(t) = \underline{o}$)

if $$3a_0 + 3b_0 + a_2 + b_2 < 0$$

and completely unstable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $\underline{x}(0) \neq \underline{o}$ there exists a $T > 0$ such that $\| \underline{x}(t) \| > \delta$ for $t > T$)

if $$3a_0 + 3b_0 + a_2 + b_2 > 0.$$

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Solution

Consider the function

$$V(\underline{x}) = x_1^2 + x_2^2 + \tfrac{1}{2}(a_1+b_3)x_1^4 - \tfrac{1}{2}(a_3+b_1)x_2^4 - (a_0-b_0)(x_1^3x_2+x_1x_2^3) + \tfrac{1}{3}(a_2+b_2)(x_1^3x_2-x_1x_2^3).$$

Then along a trajectory $\underline{x} = \underline{x}(t)$ we have

$$\dot{V}(t) = \frac{d}{dt} V(\underline{x}(t)) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} = [a_0 + b_0 + \tfrac{1}{3}(a_2+b_2)](x_1^4+x_2^4) + \text{terms of degree } 6.$$

It is obvious that V is a Lyapunov-function, i.e., V is continuously differentiable, $V(\underline{o}) = 0$, $V(\underline{x}) > 0$ in an open neighbourhood $\Omega(0 < \|x\| < \delta)$ of $\underline{x} = \underline{o}$ and $\dot{V}(t) \lessgtr 0$ for $\underline{x}(t) \in \Omega$ if $a_0 + b_0 + \tfrac{1}{3}(a_2+b_2) \lessgtr 0$.

The statements concerning stability and instability now follow from well-known theorems of Lyapunov (cf., e.g., La Salle and Lefschetz, Stability by Liapunov's Direct Method, N.Y., 1961).

Note. The basic idea of the proof of Lyapunov's theorems is the following. Let $0 < \delta_1 < \delta_2$, with sufficiently small δ_2. Then the inequalities

$$\delta_1 \leq V(\underline{x}) \leq \delta_2$$

define a closed annular subdomain Ω_1 of Ω which contains the origin inside its inner boundary. If $\dot{V} < 0$ for $\underline{x}(t) \in \Omega$ then $\dot{V} \leq -\varepsilon < 0$ as long as $\underline{x}(t) \in \Omega_1$. Hence if $\underline{x}(0) \in \Omega_1$ then certainly $V(\underline{x}(t)) < \delta_1$ for $t > (\delta_2-\delta_1)/\varepsilon$. Hence $\lim_{t \to \infty} V(\underline{x}(t)) = 0$, which implies $\lim_{t \to \infty} \underline{x}(t) = \underline{o}$. And similarly in the case of instability.