On Kleinrock's Theorem.

The following has been triggered by C.A.R. Hoare's "A general conservation law for queueing disciplines" (Information Processing Letters 2 (1972) 82-85); this article refers to L. Kleinrock, "A conservation law for queueing disciplines", Naval Research Logistics Quarterly (June 1965) 181-192, and has been submitted because it offers for Kleinrock's Theorem a proof that is simpler than the original one.

In Hoare's formulation, Kleinrock's Theorem is as follows. Assume (1) No server is idle when there is a waiting customer. (2) Arrival times and service times of both customers and servers are not affected by the decision who is to receive service at what time (i.e. they are independent of the scheduling). (3) Preemption is not used. Then

$$\sum_{i=1}^{N} W_i \cdot R_i$$

is independent of the choice of non-preemptive scheduling method. Here the customers are numbered from 1 through N, W_i is the total waiting time for customer i, and R_i is the total service time for customer i.

This note is written to point out that (1) for a single-server system the proof is trivial, and (2) for a multi-server system the theorem is not quite correct.

* *
*

Consider a schedule in which customer q is served immediately after customer p, with $W_q > R_p$. The possible decision to serve these two customers in the other order would have affected the waiting times as follows:

$$W_p, W_q := W_p + R_q , W_q - R_p \quad .$$

The products $W_p \cdot R_p$ and $W_q \cdot R_q$ are therefore increased and decreased respectively by the product of the service times, and their sum remains constant. Kleinrock's theorem follows by induction

* *
*

Consider a two-server system in which three customers, with service times $=1$, $=1$, and $=2$ respectively, arrive simultaneously. Two schedules are possible with $\Sigma\, W_i \cdot R_i = 1$ and $= 2$ respectively.

For customer i, which is served from $t0$ till $t1$ we can define a "moment of inertia" of his service as

$$\int_{t0}^{t1} t \cdot dt = \tfrac{1}{2}(t1^2 - t0^2) = (t1 - t0) \cdot \left(\frac{t0 + t1}{2}\right) =$$

$$R_i \cdot (A_i + W_i + \tfrac{1}{2} R_i) = R_i \cdot W_i + C_i$$

where A_i is the moment of arrival of customer i and $C_i = R_i \cdot (A_i + \frac{1}{2} R_i)$; our assumptions imply that C_i is independent of the scheduling. If $S(t)$ is the number of customers served at moment t, then the "moment of inertia" of total service satisfies

$$\int_0^T S(t) \cdot t \cdot dt = \sum_{i=1}^{N} R_i \cdot W_i + \sum_{i=1}^{N} C_i \quad .$$

Hence, Kleinrock's Theorem holds among schedules with the same "moment of inertia" of total service.

Acknowledgement. I am indebted to C.A.R. Hoare for having invited me to try to simplify his proof (which was already a simplification of Kleinrock's). (End of Acknowledgement.)

Plataanstraat 5
5671 AL NUENEN
The Netherlands

4 February 1981

prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow.