Fibonacci numbers and Leonardo numbers.

(The following formal derivations and computations are absolutely elementary and without scientific interest. But I am interested in some numbers and need the formulae, and learned that working on a scratch pad I make too many mistakes. Hence.)

The Fibonacci numbers are given by
$F_0 = 1 \quad F_1 = 1 \quad F_{n+2} = F_{n+1} + F_n$ (or: $F_{n+2} - F_{n+1} - F_n = 0$).
The analytical solution of a homogeneous recurrence relation like this is found by solving first the corresponding characteristic equation which one gets by "trying" an F of the form $F_n = x^n$:

(0) $\qquad x^2 - x - 1 = 0$

This equation has two different roots, which I shall denote by α and β

(1) $\quad \alpha = \frac{1}{2} + \frac{1}{2}\sqrt{5} = 1.618\,034 \qquad {}^{10}\log \alpha = 0.208\,988$

$\qquad \beta = \frac{1}{2} - \frac{1}{2}\sqrt{5} = -.618\,034$

Obvious properties are

(2) $\quad \alpha + \beta = 1 \qquad \alpha \cdot \beta = -1$

$\qquad \alpha^2 = \alpha + 1 \;, \; \alpha^3 = 2\cdot\alpha + 1 \;, \; \alpha^4 = 3\cdot\alpha + 2 \;,$ etc.

Because $\alpha \neq \beta$, α^n and β^n give rise to linearly independent sequences and <u>each</u> solution of the homogeneous recurrence relation is of the form

$$(3) \qquad F_n = X \cdot \alpha^n + Y \cdot \beta^n$$

where the constants X and Y are determined by solving the set of linear equations

$$(4) \qquad \begin{aligned} X + Y &= F_0 \\ \alpha \cdot X + \beta \cdot Y &= F_1 \end{aligned}$$

Multiplying the second equation by α we get -see (2)-
$(\alpha+1) \cdot X - Y = \alpha \cdot F_1$, and hence
$(\alpha+2) \cdot X = F_0 + \alpha \cdot F_1$, hence

$$X = \frac{F_0 + (\frac{1}{2} + \frac{1}{2}\sqrt{5}) \cdot F_1}{2\frac{1}{2} + \frac{1}{2}\sqrt{5}} = \frac{2 \cdot F_0 + (1+\sqrt{5}) \cdot F_1}{(5+\sqrt{5})} \cdot \frac{(5-\sqrt{5})}{(5-\sqrt{5})}$$

$$= \frac{1}{20}\left(10 \cdot F_0 - 2 \cdot F_0 \cdot \sqrt{5} + 4 \cdot F_1 \cdot \sqrt{5}\right) =$$

$$= \frac{5 \cdot F_0 + (2 \cdot F_1 - F_0) \cdot \sqrt{5}}{10} \qquad (5)$$

and, for reasons of symmetry

$$Y = \frac{5 \cdot F_0 - (2 \cdot F_1 - F_0) \cdot \sqrt{5}}{10} \qquad (5')$$

For the Fibonacci numbers we substitute $F_0 = F_1 = 1$ and find, according to (3)

(6) $$F_n = (\tfrac{1}{2} + \tfrac{1}{10}\sqrt{5})\cdot\alpha^n + (\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5})\cdot\beta^n$$

$$= .723\,607\cdot\alpha^n + .276\,393\cdot\beta^n$$

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We now switch to the Leonardo numbers given by $L_0 = 1$ $L_1 = 1$ $L_{n+2} = L_{n+1} + L_n + 1$. This recurrence relation is not homogeneous but —because $x = 1$ is not a root of (0)— this is only an apparent complication: $(L_{n+2}+1) = (L_{n+1}+1) + (L_n+1)$, and we immediately derive

(7) $$L_n = 2\cdot F_n - 1 \quad .$$

The nth Leonardo tree has L_n vertices. The (n+2)th Leonardo tree is a binary tree, of which the (n+1)th and the nth Leonardo tree are the two subtrees. A number that is perhaps of some interest is the distance from the root summed over the vertices of the nth Leonardo tree. Denoting this quantity by K_n we derive from the definition

(8) $$K_{n+2} = (K_{n+1} + L_{n+1}) + (K_n + L_n)$$

or

$$(K_{n+2} - 2) = (K_{n+1} - 2) + (K_n - 2) + (L_{n+1}+1) + (L_n+1)$$

or with

(9) $$K_n = 2\cdot(H_n + 1) \quad \text{or} \quad H_n = (K_n - 2)/2$$

(10) $$H_{n+2} = H_{n+1} + H_n + F_{n+2}$$

Solving (10) for F_{n+2} and taking the recurrence relation for the F's into account one finds that the H's satisfy a homogeneous linear recurrence relation with $(x^2-x-1)^2=0$ as characteristic equation. (This is a special case of a more general theorem of which I was not aware.) Hence the general form of H_n is

(11) $\quad H_n = (a+n\cdot A)\cdot\alpha^n + (b+n\cdot B)\cdot\beta^n$

where the constants $a, A, b,$ and B are determined by solving — see (2) —

(12)
$$\begin{array}{rcrcrcrcl}
a & + & & & b & & & = & H_0 \\
\alpha\cdot a & + & \alpha\cdot A & + & \beta\cdot b & + & \beta\cdot B & = & H_1 \\
(\alpha+1)\cdot a & + & (2\alpha+2)\cdot A & + & (\beta+1)\cdot b & + & (2\cdot\beta+2)\cdot B & = & H_2 \\
(2\cdot\alpha+1)\cdot a & + & (6\cdot\alpha+3)\cdot A & + & (2\cdot\beta+1)\cdot b & + & (6\cdot\beta+3)\cdot B & = & H_3
\end{array}$$

We eliminate a and b with

(13) $\quad y_0 = H_2 - H_1 - H_0 \quad , \quad y_1 = H_3 - H_2 - H_1$

$$\begin{array}{rcrcl}
(\alpha+2)\cdot A & + & (\beta+2)\cdot B & = & y_0 \\
(3\cdot\alpha+1)\cdot A & + & (3\cdot\beta+1)\cdot B & = & y_1
\end{array} \quad ,$$

which leads to

$$\begin{array}{rcrcl}
5\cdot A & + & 5\cdot B & = & z_0 \\
\alpha\cdot(5\cdot A) & + & \beta\cdot(5\cdot B) & = & z_1
\end{array} \qquad \text{with}$$

(14) $\quad z_0 = 3\cdot y_0 - y_1 \quad , \quad z_1 = 2\cdot y_1 - y_0$

This set of equations is of the same form as (4). Hence we have

(15) $\quad A = \dfrac{5\cdot z_0 + (2\cdot z_1 - z_0)\cdot\sqrt{5}}{50}$

(15') $\quad B = \dfrac{5\cdot z_0 - (2\cdot z_1 - z_0)\cdot\sqrt{5}}{50}$

We can eliminate A and B from (12) with

$$y_3 = -2\cdot H_3 + 3\cdot H_2 + 6\cdot H_1 - H_0$$

$$\begin{aligned} 5\cdot a \quad + \quad 5\cdot b \quad &= 5\cdot H_0 \\ \alpha\cdot(5\cdot a) + \beta\cdot(5\cdot b) &= y_3 \end{aligned}$$

which is again of the form (4). Eliminating y_3, we get

(16) $\quad a = \dfrac{25\cdot H_0 + (-4\cdot H_3 + 6\cdot H_2 + 12\cdot H_1 - 7\cdot H_0)\cdot\sqrt{5}}{50}$

(16') $\quad b = \dfrac{25\cdot H_0 - (-4\cdot H_3 + 6\cdot H_2 + 12\cdot H_1 - 7\cdot H_0)\cdot\sqrt{5}}{50}$

Let us apply these formulae with the numerical values of K_0, K_1, K_2, K_3 and then check them against the numerical value of K_4. (It is a long time ago since I made my last check.)

n:	L_n:	K_n:	H_n
0	1	0	−1
1	1	0	−1
2	3	2	0
3	5	6	2
4	9	16	7

From (13): $y_0 = 2 \quad y_1 = 3$

From (14): $z_0 = 3 \quad z_1 = 4$

From (15):

$$A = \frac{15 + 5\cdot\sqrt{5}}{50} = \frac{3+\sqrt{5}}{10}$$

$$B = \frac{3-\sqrt{5}}{10}$$

From (16) $$a = \frac{-25 + (-8 - 12 + 7)\cdot\sqrt{5}}{50} = \frac{-25 - 13\cdot\sqrt{5}}{50}$$

$$b = \frac{-25 + 13\cdot\sqrt{5}}{50} \quad .$$

In order to check these values for $n=4$ we compute

$$(a + 4\cdot A)\cdot\alpha^4 = \frac{1}{50}\cdot\left(-25 - 13\cdot\sqrt{5} + 60 + 20\cdot\sqrt{5}\right)\cdot(3\cdot\alpha + 2)$$

$$= \frac{1}{100}(35 + 7\cdot\sqrt{5})(7 + 3\cdot\sqrt{5}) = \frac{1}{100}\left(245 + 105 + 154\cdot\sqrt{5}\right) =$$

$3\frac{1}{2} + \frac{154}{100}\cdot\sqrt{5}$. This is OK and that is very encouraging.

We are now in a position to compute the asymptotic behaviour of the average distance from the root

$$\frac{K_n}{L_n} = \frac{2\cdot H_n + 2}{2\cdot F_n - 1} \rightarrow \frac{H_n}{F_n} \rightarrow \frac{\frac{1}{10}(3+\sqrt{5})}{\frac{1}{10}(5+\sqrt{5})}n = \frac{5+\sqrt{5}}{10}\cdot n$$

With N the number of points, the average distance grows as $\frac{5+\sqrt{5}}{10}\cdot {}^{\alpha}\log N = .723\,607\cdot {}^{\alpha}\log N$, a growth rate I would like to compare to the one of the completely balanced binary tree. The number of nodes in the nth binary tree equals $2^{n+1} - 1$. The sum over its nodes of their distances from the root is $(\underline{S}\, i: 0 \leq i \leq n: i \cdot 2^i) = (n-1) \cdot 2^{n+1} + 2$. With N the number of points, the dominant term of the growth rate of the average distance from the root is therefore ${}^2\log N$. For the Leonardo trees it is $.723\,607 \cdot {}^{\alpha}\log N = 1.042\,296 \cdot {}^2\log N$. The ratio is —as was to be expected— larger than 1, but only very little so. (I am not convinced of the relevance of the notion "average distance from the root"; it has the advantage that the above estimations can be derived by elementary means.)

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We know that, with a given number, taking away the largest possible Leonardo number and repeating this process on the remainder, we decompose the given number, x say, in the minimum number f(x) of Leonardo numbers. What is the average value of f(x) when x ranges over the first N natural numbers?

Defining $D_i = (\underline{S}\, x: 0 \leq x < L_{i+1}: f(x))$, we have

$D_0 = 0$, $D_1 = 3$, $D_{n+2} = D_{n+1} + D_n + L_{n+1} + 2$,

hence $D_2 = 6$, $D_3 = 14$, $D_4 = 27$, etc.

This is the moment I am going to reap the fruit of (13), (14), and (15). With $H_n = D_n + 1$ we have $H_{n+2} = H_{n+1} + H_n + 2 \cdot F_{n+1}$, and (11) is applicable. We have $H_0 = 1$, $H_1 = 4$, $H_2 = 7$, and $H_3 = 15$. From (13) $y_0 = 2$, $y_1 = 1$; from (14) $z_0 = 2$, $z_1 = 6$, and from (15) $A = \frac{1}{50}(10 + 10 \cdot \sqrt{5}) = \frac{1}{5}(1 + \sqrt{5})$. Hence the dominant term of H_n (and D_n) is $\frac{1}{5}(1+\sqrt{5}) \cdot n \cdot \alpha^n$. The leading term of L_{n+1} ($= 2 \cdot F_{n+1} - 1$) is $\frac{1}{5}(5+\sqrt{5}) \cdot \alpha^{n+1} = \frac{1}{10}(1+\sqrt{5})(5+\sqrt{5}) \cdot \alpha^n$.

The growth rate of the average value of $f(x)$ is that of H_n / L_{n+1}, i.e. $\frac{2}{(5+\sqrt{5})} \cdot n = \frac{1}{10}(5-\sqrt{5}) \cdot n = .276\,393 \cdot {}^{\alpha}\log N$.

Analogously to the perfectly balanced binary trees we can replace the Leonardo numbers by $B_n = 2^{n+1} - 1$. Let $f'(x)$ be the minimum number of B's with sum x and let $C_n = (\underline{S} : 0 \le x \le B_n : f'(x))$. We find $C_n = (n+1) \cdot 2^n$. In this case the growth rate of the average value of $f'(x)$ is —not surprisingly— $C_n / B_n = \frac{1}{2}(n+1) = \frac{1}{2} \cdot {}^{2}\log N = {}^{4}\log N$. Comparing this with the case of the Leonardo numbers

$$.276\,393\ {}^{\alpha}\log N = .796\,243 \cdot {}^{4}\log N$$

and this time the ratio is markedly smaller than 1.

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12th July 1981
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