Two cheers for equivalence

Let us consider the operators $\neg$, $\vee$, and $\equiv$; $\neg$ is a unary operator, $\vee$ and $\equiv$ are symmetric and associative operators defined on bags of at least 2 operands. For the latter two we adopt the usual infix notation; the three operators have been listed in the order of decreasing syntactic binding power.

In the presence of $P \equiv R$ a new formula may be formed by replacing in an existing formula one or more occurrences of P by R . (Leibniz's Rule.)

Axiom 0 $\quad P \equiv P \equiv Q \vee \neg Q$.

Parsing this $(P \equiv P) \equiv (Q \vee \neg Q)$ we see that $Q \vee \neg Q$ may be replaced by $P \equiv P$, which does not depend on Q ! This suggests to introduce the abbreviation

Abbr. 0 $\quad Q \vee \neg Q \equiv \text{black}$

(where "black" may be viewed as a constant).

Applying Leibniz's Rule to the above two formulae we generate

Theorem 0 $\quad P \equiv P \equiv \text{black}$.

Parsing this as $P \equiv (P \equiv \text{black})$, and applying Leibniz's Rule we see that the suffix $\equiv \text{black}$ can be removed from a formula that ends on it; we are also free to add it to an existing formula. So we derive from Theorem 0 and Abbr.0 respectively

Theorem 1 $\quad P \equiv P$.

Theorem 2 $\quad Q \vee \neg Q$.

We now add

Axiom 1 $\quad P \vee \neg Q \equiv P \vee Q \equiv P$.

Substituting P for Q in Axiom 1 yields

$$\text{black} \equiv P \vee P \equiv P$$

yielding with Theorem 0

<u>Theorem 3</u> $\quad P \vee P \equiv P \quad .$

Applying Theorem 3 to Abbr. 0 yields

$$Q \vee Q \vee \neg Q \equiv black \quad ,$$

yielding with Abbr. 0

<u>Theorem 4</u> $\quad P \vee black \equiv black \quad .$

Substituting black for Q in Axiom 1 yields

$$P \vee \neg black \equiv P \vee black \equiv P$$

and by application of Theorems 4 and 0

<u>Theorem 5</u> $\quad P \vee \neg black \equiv P \quad .$

Substitution of ¬black for P in Axiom 1 yields

$$\neg black \vee \neg Q \equiv \neg black \vee Q \equiv \neg black$$

and by applying Theorem 5 twice we get

<u>Theorem 6</u> $\quad \neg Q \equiv Q \equiv \neg black \quad .$

Substitution of ¬Q for Q yields

$$\neg\neg Q \equiv \neg Q \equiv \neg black$$

and from the latter two we get with Leibniz's Rule

<u>Theorem 7</u> $\quad \neg\neg Q \equiv Q \quad .$

Substituting in Axiom 1 $P \vee Q$ for Q, we get

$$P \vee \neg(P \vee Q) \equiv P \vee P \vee Q \equiv P$$

yielding with Theorem 3

$$P \vee \neg(P \vee Q) \equiv P \vee Q \equiv P \quad .$$

Confronting this with Axiom 1, we get

<u>Theorem 8</u> $\quad P \vee \neg(P \vee Q) \equiv P \vee \neg Q \quad .$

Substituting $P \equiv Q$ for Q in Theorem 6 we get

$$\neg(P \equiv Q) \equiv P \equiv Q \equiv \neg \text{black}$$

and applying Theorem 6 once more we generate

<u>Theorem 9</u> $\quad \neg(P \equiv Q) \equiv P \equiv \neg Q$.

With Theorems 1 and 6 we generate in succession

$$\neg \text{black} \equiv \neg \text{black} \equiv \text{black}$$

$$P \equiv \neg P \equiv R \equiv \neg R \equiv Q \equiv \neg Q \quad , \text{ i.e.}$$

<u>Theorem 10</u> $\quad P \equiv Q \equiv R \equiv \neg P \equiv Q \equiv \neg R$.

Substitution of $\neg P$ for P in Axiom 1 yields

$$\neg P \vee \neg Q \equiv \neg P \vee Q \equiv \neg P \quad .$$

With Theorem 10 this yields

$$\neg(\neg P \vee \neg Q) \equiv \neg P \vee Q \equiv P \quad .$$

and with

<u>Abbr. 1</u> $\quad \neg(\neg P \vee \neg Q) \equiv P \wedge Q$.

<u>Theorem 11.</u> $\quad P \wedge Q \equiv \neg P \vee Q \equiv P$.

In the sequel, appeals to Abbr. 1 and Theorem 7 wil often be summarized by referring to the Law of de Morgan.

Substitution of $P \wedge Q$ for Q in Axiom 1 yields

$$P \vee \neg(P \wedge Q) \equiv P \vee (P \wedge Q) \equiv P \quad .$$

With de Morgan's Law

$$P \vee \neg P \vee \neg Q \equiv P \vee (P \wedge Q) \equiv P \quad .$$

With Abbr. 0, Theorems 4 and 0 we generate

<u>Theorem 12</u> $\quad P \vee (P \wedge Q) \equiv P$.

Interchanging in Axiom 1 P and Q gives

$$Q \vee \neg P \equiv P \vee Q \equiv Q$$

which yields with Axiom 1

<u>Theorem 13</u> $P \equiv Q \equiv Q \vee \neg P \equiv P \vee \neg Q$.

Applying Theorem 12 we derive from Theorem 13

$$P \equiv Q \equiv Q \vee (Q \wedge \neg P) \vee \neg P \equiv P \vee \neg Q \quad ;$$

with de Morgan's Law

$$P \equiv Q \equiv \neg (P \vee \neg Q) \vee (Q \vee \neg P) \equiv P \vee \neg Q$$

and applying Theorem 11, we generate

<u>Theorem 14</u> $P \equiv Q \equiv (P \vee \neg Q) \wedge (Q \vee \neg P)$.

<u>Note</u> Theorem 14 corresponds to the Hilbert-Ackermann definition of equivalence. (End of Note.)

From Theorem 4 we derive

$$Q \vee R \vee \text{black}$$

from which we generate with Abbr. 0

$$Q \vee P \quad \vee \quad R \vee \neg P$$

which yields with Theorem 8 (twice)

$$Q \vee \neg (Q \vee \neg P) \vee R \vee \neg (R \vee P)$$

which yields with de Morgan's Law

<u>Theorem 15</u> $(\neg Q \wedge P) \vee (\neg R \wedge \neg P) \vee Q \vee R$.

<u>Note</u> Theorem 15 corresponds to the last axiom of Hilbert-Ackermann

$$(P \Rightarrow Q) \Rightarrow ((P \vee R) \Rightarrow (Q \vee R)) \quad .$$

(End of Note).

Now comes a trivial section that I shall only indicate. With

Abbr. 2 $\quad \neg\,\text{black} \equiv \text{white}$

we leave it to the reader to generate — mostly with de Morgan's Law — all sorts of useful theorems such as

$$Q \wedge \neg Q \equiv \text{white}$$

$$P \wedge P \equiv P$$
$$P \wedge \text{white} \equiv \text{white}$$
$$P \wedge \text{black} \equiv P$$
$$P \wedge (\neg P \vee Q) \equiv P \wedge Q$$
$$P \wedge (P \vee Q) \equiv P \quad .$$

So far I did not succeed in generating, say

$$(P \equiv Q) \wedge (P \equiv R) \equiv (P \equiv Q) \wedge (Q \equiv R)$$

or the distributivity of $\wedge$ and $\vee$. I have tried whether I could modify my axioms — currently, none of them contains three variables — but did not succeed. The obvious alternative is the generalization of Leibniz's Rule: if Q could be generated in the additional presence of P, we allow ourselves to generate $\neg P \vee Q$.

Since I don't want to become a logician I had better stop; in any case I have had my fun.

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24 October 1982

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