On maximizing a product

The problem is to construct a set of positive integers with given sum such that their product is as large as possible.

The first observation is that we need not include integers ≥ 4. The reason is that such an integer k can be replaced, without changing the sum, by the two integers 2 and $k-2$, thereby replacing in the product a factor k by a factor $2\cdot k-4$ and that we don't lose if $2\cdot k - 4 \geq k$, i.e. if $k \geq 4$.

The second observation is that 1 only needs to be included if it is the only integer in the set, i.e. if the given sum equals 1. In all other cases we can increase the product without changing the sum by removing the 1 (which does not contribute to the product) and increasing one of the other integers by 1.

In the general case this leaves us with a set of 2's and 3's. The final observation is that we don't need to include 3 or more 2's: $2+2+2 = 3+3$ but $2\cdot 2\cdot 2 < 3\cdot 3$.

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We solved this problem a number of years ago (and were not the first to do so), but in my memory the argument was less clean than the above. It was more or less polluted by the circumstance that the answer is not unique if the given sum is of the form $3\cdot n+4$: two 2's may be replaced by one 4.

Another way of complicating the first observation is to treat even and odd values separately, e.g. comparing for $k=2\cdot x$ the value $2\cdot x$ with x^2 and for $k=2\cdot x+1$ the value $2\cdot x+1$ with $x\cdot(x+1)$.

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