A monotonicity argument

In the following, square brackets denote universal quantification over x and y ; f is a function of two arguments such that

$$[x < y \;\Rightarrow\; fxy < fyx] \quad . \qquad (0)$$

From the above, we derive the following results.

Since $p < q \equiv q > p$, from (0):

$$[y > x \;\Rightarrow\; fyx > fxy] \quad .$$

From the above, by renaming the dummies:

$$[x > y \;\Rightarrow\; fxy > fyx] \quad . \qquad (1)$$

From $[x = y \Rightarrow fxy = fyx]$, by taking the term-wise disjunction with (0) and (1) respectively:

$$[x \le y \;\Rightarrow\; fxy \le fyx] \qquad (2a)$$

$$[x \ge y \;\Rightarrow\; fxy \ge fyx] \quad . \qquad (3a)$$

By taking the counterpositives of (1) and (0) respectively – which to all intents and purposes amounts to doing nothing –

$$[fxy \le fyx \;\Rightarrow\; x \le y] \qquad (2b)$$

$$[fxy \ge fyx \;\Rightarrow\; x \ge y] \quad . \qquad (3b)$$

From (2) and (3) respectively:

$$[x \le y \;\equiv\; fxy \le fyx] \qquad (4)$$

$$[x \ge y \;\equiv\; fxy \ge fyx] \quad . \qquad (5)$$

From (4) and (5), by taking the term-wise conjunction:

$$[x = y \;\equiv\; fxy = fyx] \quad . \qquad (6)$$

By negating both sides of (5) and (4) respectively:

$$[x < y \;\equiv\; fxy < fyx] \qquad (7)$$
$$[x > y \;\equiv\; fxy > fyx] \quad . \qquad (8)$$

It is nice to see how the five equivalences (4) through (8) follow from the single implication (0). Note that, since for reasons of symmetry (0) and (1) are equivalent, all formulae could also have been derived from (1).

We draw attention to the fact that we did not use that f depends on both its arguments: if f only depends on its first argument, (0) postulates that f is an increasing function, but if f only depends on its second argument, (0) postulates that f is a decreasing function of that argument.

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The above grew out of our dissatisfaction with the way we were introduced to Euclidean geometry. With congruences, two theorems were proved separately, viz.

"An isosceles triangle has two equal angles." and
"A triangle with two equal angles is isosceles." .

Prior to that it had already been shown that opposite to the larger angle lies the larger side, in the jargon: $\alpha > \beta \Rightarrow a > b$. This is a statement of the form (1); for the proofs of the theorems, congruences are not needed at all, since the theorems are subsumed in the conclusion corresponding to (6): $\alpha = \beta \equiv a = b$.

We had similar experiences with the two theorems

"An isosceles triangle has two angle bisectors of equal length." and "A triangle with two angle bisectors of equal length is isosceles." .

Thanks to all the above, both theorems are subsumed in

"In a triangle, the larger angle has the shorter bisector.",

a theorem that can be proved in a variety of ways. For instance, in a triangle with sides of lengths a, b, and c, the square of the length of the bisector of the angle opposite to a equals

$$b \cdot c \cdot \left(1 - \left(\frac{a}{b+c}\right)^2\right) .$$

This expression is an increasing function of b and a decreasing function of a. It is not difficult to show that an f that is an increasing function of its first argument and a decreasing function of its second argument satisfies (0).

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