# On the design of a simple proof for Morley's Theorem

The general opinion —with which I concur— is that Frank Morley's Theorem about the angle trisectors of a triangle is a geometrical curiosity that is of historical interest at best. Independently of the (in)significance of the theorem proved by it, a proof may deserve our attention, for instance, by virtue of its structure, its simplicity, or its brevity. For the proof to be shown below, such a claim can be made: requiring no auxiliary lines or points, it is so simple that the theorem's late discovery (1899) and the elapsed decade before the first proofs were published (1909) become the more striking. When I found this proof years ago, I was only too willing to ascribe that discovery to my great ingenuity and all that. The purpose of this note, however, is to show how, in the mean time, the art and science of proof design have advanced to a stage in which the design of such proofs has almost become a routine exercise, requiring the usual care in arrangement and notation, but a minimum of invention.

Consider a plane figure —consisting of 6 distinct points and 12 straight line segments— of the following configuration —
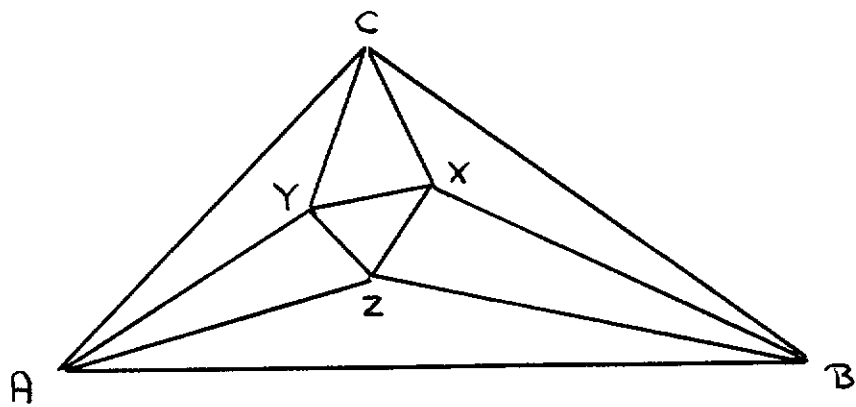
Morley's Theorem states that △XYZ is equilateral if, in this figure, the angles of △ABC are trisected.

Note that there are all sorts of nonredundant ways of defining that a triangle is equilateral, such as
(0) its edges are of equal length
(1) its angles are of equal size
(2) it is isosceles, and its top angle equals 60°.
Definition (1) makes it clear that Morley's Theorem can be formulated in terms of angles only, i.e. that in the above figure we can isolate size from shape.

Let us call the above configuration with the angles of △ABC trisected a Morley Figure. The simplest way of giving the shape of a Morley Figure independently of its size is to give the angles of △ABC. The simplest way of giving the size of a Morley Figure independently of its shape is to give the size of △XYZ.

This is a very nice disentanglement. To do

justice to it, we propose to prove Morley's Theorem by showing

For any triple (α, β, γ) of positive angles whose sum equals $60°$ and for any equilateral triangle, one can erect on the three sides of the equilateral triangle triangles with top angles α, β, γ respectively so that the angles of the triangle formed by their tops are trisected.

The existence of these three triangles will be demonstrated by showing how to erect them. To this end we shall derive how to erect them. We exploit the cyclic symmetry (which we have been careful not to destroy) in order to avoid repetition, and focus our attention on side AB
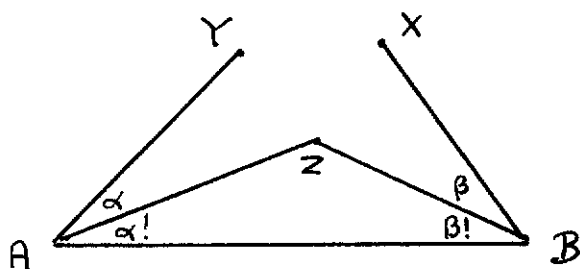
We have marked with $\alpha!$ and $\beta!$ the two target sizes, with α and β the two given sizes, and in view of the many alternative definitions we have left open how we are going to take into account that $\triangle XYZ$ is equilateral. In the above picture, we can draw one immediate conclusion from the target sizes: $\angle AZB = 180° - \alpha - \beta$. Because the remainder of the picture has to be

taken into account, we have to conclude that its full complement equals $180° + \alpha + \beta$. Now it stands to reason to draw XZ and YZ and to take into account that $\angle XZY = 60°$
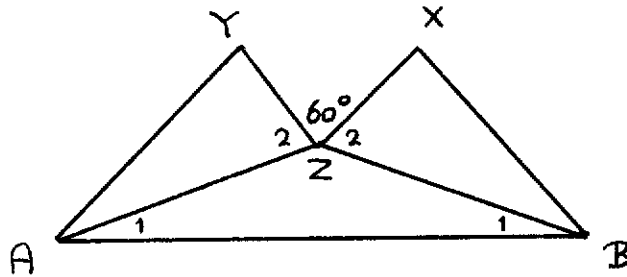

From $\angle XZY = 60°$ we thus conclude

$(\alpha + \beta =$ the sum of the angles marked 1$) \equiv$
$(120 + \alpha + \beta =$ the sum of the angles marked 2$)$.

A prescribed sum of two values gives a one-dimensional degree of freedom, for which we now have to find the most suitable parameterization. For the angles marked 1, the simplest parameterization is
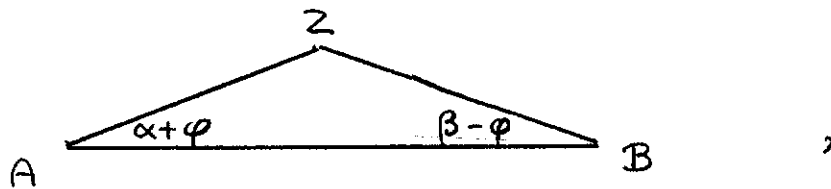
since that translates our target into $\varphi = 0$. For reasons of symmetry it stands to reason to split $120° + \alpha + \beta$ into $60° + \alpha + \psi$ and $60° + \beta - \psi$, but which is which? Because of $\alpha + \beta + \gamma = 60°$, we get the nicest fit if we allocate $60° + \alpha + \psi$ in the triangle with top β. Thus

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we find ourselves contemplating the following parameterized figure:
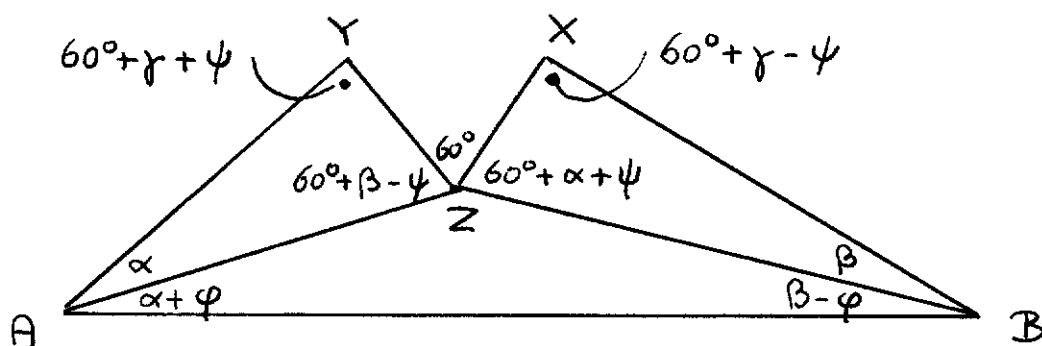
Were we to complete it, we would find in clockwise direction $60°+\alpha+\psi$, $60°+\beta-\psi$, $60°+\gamma+\psi$, $60°+\alpha-\psi$, $60°+\beta+\psi$, $60°+\gamma-\psi$. Our remaining duty is to show that ψ can be chosen in such a way that $\varphi=0$ follows.

Looking at our last diagram, we should realize that, so far, we have used from geometry no more than the fact that the angles of a triangle add up to $180°$. We should be willing to use a little bit more so as to be able to exploit that the three triangles share two sides. Parameter φ (for which $=0$ has to be concluded) occurs only twice in the above picture, more precisely, in $\triangle AZB$ opposite to the shared sides. So we are looking for a fact from geometry that links angles of a triangle to their opposite sides. Here the Rule of Sines is supposed to come to the reader's

mind. Furthermore, of the fact that $\triangle XYZ$ is equi-lateral, so far we used only $\angle XZY = 60°$ and thus can expect to have to use $XZ = YZ$.

So, let us start with the expression in φ that we can manipulate with the Rule of Sines

$$\frac{\sin(\alpha + \varphi)}{\sin(\beta - \varphi)}$$

$=$ {Rule of Sines in $\triangle AZB$}

$$\frac{BZ}{AZ}$$

$=$ {Rule of Sines in $\triangle BZX / \triangle AZY$ so as to introduce XZ / YZ}

$$\frac{\sin(60° + \gamma - \psi) \cdot XZ / \sin(\beta)}{\sin(60° + \gamma + \psi) \cdot YZ / \sin(\alpha)}$$

$=$ { $XZ = YZ$}

$$\frac{\sin(60° + \gamma - \psi) \cdot \sin(\alpha)}{\sin(60° + \gamma + \psi) \cdot \sin(\beta)} \qquad ,$$

In summary

(3) $\qquad \dfrac{\sin(\alpha + \varphi)}{\sin(\beta - \varphi)} = \dfrac{\sin(60° + \gamma - \psi)}{\sin(60° + \gamma + \psi)} \cdot \dfrac{\sin(\alpha)}{\sin(\beta)}$

Now the question has become: can we substitute in (3) such a value for ψ that $\varphi = 0$ follows? Substituting into (3) $\varphi := 0$ yields

(4) $\qquad \dfrac{\sin(60° + \gamma - \psi)}{\sin(60° + \gamma + \psi)} = 1 \qquad ,$

a solvable equation (eg. $\psi = 0$). Substituting (4) into (3),

we get

$$\sin(\beta) \cdot \sin(\alpha+\varphi) = \sin(\alpha) \cdot \sin(\beta-\varphi)$$

= {addition formula for sin}

$$\sin(\beta) \cdot (\sin(\alpha) \cdot \cos(\varphi) + \cos(\alpha) \cdot \sin(\varphi)) =$$
$$\sin(\alpha) \cdot (\sin(\beta) \cdot \cos(\varphi) - \cos(\beta) \cdot \sin(\varphi))$$

= {algebra}

$$(\sin(\beta) \cdot \cos(\alpha) + \sin(\alpha) \cdot \cos(\beta)) \cdot \sin(\varphi) = 0$$

= {addition formula for sin}

$$\sin(\alpha+\beta) \cdot \sin(\varphi) = 0$$

= $\{ 0 < \alpha+\beta < 60° \}$

$$\sin(\varphi) = 0$$

= $\{ \alpha+\varphi \geqslant 0 \Rightarrow \varphi > -60°; \ \beta-\varphi \geqslant 0 \Rightarrow 60° > \varphi \}$

$$\varphi = 0 \quad ,$$

which concludes the proof. (To solve the last equation with the addition formula for the sine, rather than via a monotonicity argument, was suggested to me by Ambuj Kumar Singh.)

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For a traditional presentation of a geometric proof, see [o] – Coxeter amazingly contrasts "a trigonometrical proof" with "an elementary proof" – ; for a compact presentation of the above trigonometrical proof, see [1] (where the proof is reduced to almost a one-liner). This note is a critical comment on such proof presentations, which are correct, but rather uninstructive.

Coxeter starts his proof with "Much trouble is

experienced if we try a direct approach, but the difficulties disappear if we work backwards, beginning with an equilateral triangle and building up a general triangle which is afterwards identified with the given triangle ABC.". Coxeter at least mentions the "trouble" — be it as a fact learned from unanalysed experience gained from trying to prove Morley's Theorem — ; Dijkstra does not even mention the trouble and just announces "We start in our proof not with the arbitrary triangle, but with the equilateral one.".

They both pull the same big rabbit out of their mathematical hat. Above we showed how this rabbit can be avoided. In order not to interrupt the flow of the presentation, we mentioned the catchword "disentanglement" — see [2] — ; here is the place to point out that it refers to a general principle. The idea is to write in particular the information to be used as a conjunction of as many independent terms as possible because this is a precondition for subsequently being able to indicate precisely what is appealed to where. (If a datum can be factored differently — such as "$\triangle XYZ$ is equilateral" — the advice is to list the options and to postpone the choice until it emerges which one is appropriate.) The decision not to fix the size of the Morley Figure by , say, the length of AB rests on the principle — see [2] — not to destroy symmetry lightly.

The second big rabbit that [0] and [1] share
is the choice of very special angles that new lines
make with the edges of the equilateral triangle.
The above analysis, based on angles alone, shows
that we are essentially left with one degree of
freedom, viz. the relative orientation of the two
triangles; hence ψ. The magician's proof imme-
diately substitutes for ψ its proper value (so
that ψ need not be mentioned at all), thus sug-
gesting that without such spark of genius one
cannot find such proofs. It is good to know
that a whole class of such sparks are superfluous:
in case of freedom, capture that freedom by the
introduction of as yet undetermined parameters,
which, if not truly arbitrary, will be later con-
strained by the needs of the subsequent proof.

The third big rabbit in [0] and [1] is the
absence of any motivation for the choice of
mathematical fact to exploit. In [0], out of the
blue Coxeter refers to a theorem about the incenter
of a triangle that I was not aware of. In [1],
Dijkstra writes equally unmotivated "Using the
rule of sines three times (in triangles AXB,
BXZ, and AXY) we deduce [....]", leaving the
reader in awe for the miracle how subsequently
all nasty terms cancel out. If we wish to teach
our students how to design proofs we should
teach them how to let their logical needs guide

them through their mathematical knowledge.

Finally I would like to point out that the above design has been driven by different concerns. Using that AZ is a shared edge of two triangles was a logical necessity, our choice of parameters was a matter of expedience. Instead of angles $\alpha + \varphi$ and $\beta - \varphi$ we could have introduced φ' and $\alpha + \beta - \varphi'$; it would have uglified the later development.

[0] Coxeter, H.S.M., "Introduction to Geometry", John Wiley & Sons, Inc., New York, 1969, pp. 23-25.

[1] Dijkstra, Edsger W., "Selected Writings on Computing: A Personal Perspective", Springer-Verlag New York Inc., 1982, pp. 182-183.

[2] Gasteren, A.J.M. van, "On the shape of mathematical arguments", Ph.D. Thesis, 20 December 1988, Technical University Eindhoven.

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