On the selection of dummies: a notational freedom

The following theorem is so well-known that it is often used subconsciously.

Theorem 0 For all J, W, P, X of the appropriate types

$$(0) \Rightarrow [(1) \equiv (2)] \qquad \text{with}$$

(0) $w \in W \equiv (\underline{E}j: j \in J: [w = X.j])$

(1) $(\underline{E}j: j \in J: P.(X.j))$

(2) $(\underline{E}w: w \in W: P.w)$.

Proof We observe

$$
\begin{aligned}
& (2) \\
= & \quad \{(0)\} \\
& (\underline{E}w: (\underline{E}j: j \in J: [w = X.j]): P.w) \\
= & \quad \{\text{trading and unnesting}\} \\
& (\underline{E}w, j:: j \in J \wedge [w = X.j] \wedge P.w) \\
= & \quad \{\text{trading and nesting}\} \\
& (\underline{E}j: j \in J: (\underline{E}w: [w = X.j]: P.w)) \\
= & \quad \{\text{one-point rule}\} \\
& (1) \quad .
\end{aligned}
$$

(End of Proof.)

The analogous theorem, obtained by replacing in (1) and (2) the existential quantifications by universal ones, follows from the above.

An advantage of (2) over (1) is that it is a little bit simpler; with more expressions of type (1) or $P.(X.j)$ written as an expression with several occurrences of $X.j$, the advantage can be quite marked.

The above admits a straightforward generalization to a function P defined on pairs:

$$(3) \Rightarrow [(4) \equiv (5)] \quad \text{with}$$

(3) $\quad w \in W \;\equiv\; (\underline{E}j: j \in J: [w = (X.j, Y.j)])$ for all w

(4) $\quad (\underline{E}j: j \in J: P.(X.j, Y.j))$

(5) $\quad (\underline{E}w: w \in W: P.w)$,

and, comparing (4) and (5), one might think the gain greater, but this is an illusion if $P.(X.j, Y.j)$ is written out as an expression in $X.j$ and $Y.j$. Using the selectors π and μ , defined by

(6) $\quad w = (\pi.w, \mu.w)$ for all pairs w ,

expression (5) is then more honestly rendered by

(7) $\quad (\underline{E}w: w \in W: P.(\pi.w, \mu.w))$,

which is of exactly the same structure as (4): (7) is (4) with $j, J, X, Y := w, W, \pi, \mu$.

This time, however, most of the lost advantage can be regained by increasing the number of dummies, thanks to the following theorem.

Theorem 1 $\quad [(5) \equiv (8)]$ with

(8) $\quad (\underline{E}\, p,q : (p,q) \in W : P.(p,q))$.

Proof We observe

$$
\begin{array}{ll}
 & (8) \\
= & \{\text{one-point rule}\} \\
 & (\underline{E}\, p,q : (\underline{E}\, w : [(p,q) = w] : w \in W) : P.(p,q)) \\
= & \{\text{trading and unnesting}\} \\
 & (\underline{E}\, p,q,w :: [(p,q) = w] \wedge w \in W \wedge P.(p,q)) \\
= & \{(6)\} \\
 & (\underline{E}\, p,q,w :: [(p,q) = (\pi.w, \mu.w)] \wedge w \in W \wedge P.(p,q)) \\
= & \{\text{pair-forming, see Note below}\} \\
 & (\underline{E}\, p,q,w :: [p = \pi.w] \wedge [q = \mu.w] \wedge w \in W \wedge P.(p,q)) \\
= & \{\text{trading and nesting}\} \\
 & (\underline{E}\, w : w \in W : (\underline{E}\, p : [p = \pi.w] : (\underline{E}\, q : [q = \mu.w] : P.(p,q)))) \\
= & \{\text{one-point rule, twice}\} \\
 & (\underline{E}\, w : w \in W : P.(\pi.w, \mu.w)) \\
= & \{(6)\} \\
 & (5) \quad .
\end{array}
$$

Note From pair-forming we have used the axioms

$$[(a,b) = (c,d) \equiv (a = c,\ b = d)]$$
$$[(a,b)] = [a] \wedge [b] \ .$$

(End of Note.)
(End of Proof.)

Since often the range, being constant, is not repeated during longer calculations, we should compare $(\underline{E}j :: P.(X.j, Y.j))$ with $(\underline{E}p,q :: P.(p,q))$; the gain can be significant.

Note that the introduction of two dummies p,q instead of the one w is quite similar to the template for formal parameters as, for instance, in SASL. The function inc that increases each element of a sequence by 1 has two alternative definitions:

inc.w = 1+ head.w : inc.(tail.w)
inc.(p:q) = 1+p : inc.q

* * *

The theorems are familiar, the proofs are straightforward. This note has been written to draw attention to the notational freedom they imply. Had Carel S. Scholten and I been more aware of that freedom and its significance, several of the proofs in our book would have been more crisp.

Inks Lake State Park, 6 October 1989

prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA