On a proof of Kaplansky's Theorem

Th. "In a ring with identity, an element without a left inverse but with at least one right inverse has infinitely many right inverses."

* * *

Let b be an element of the ring. Denoting "multiplication" by juxtaposition, the left inverses of b are the solutions of the equation

(0) $x: (xb = 1)$;

we assume that (0) has no solutions. The right inverses of b are the solutions of the equation

(1) $x: (bx = 1)$;

let a_k for $0 \le k < N$ be N distinct solutions of (1), i.e.

(2) $ba_k = 1$

(3) $a_h = a_k \Rightarrow h = k$.

We shall show that if $N > 0$, (1) has at least $N+1$ solutions. From now on, we assume $N > 0$, i.e. the existence of a_0.

Consider the N quantities c_k given by

(4) $c_k = a_0 + 1 - a_k b$ for $0 \le k < N$

<u>Lemma 0</u> Each c_k is a right inverse of b.

<u>Proof</u> We observe for any k, $0 \le k < N$

$$
\begin{array}{ll}
 & b\,c_k \\
= & \quad \{(4)\} \\
 & b\,(a_0 + 1 - a_k b) \\
= & \quad \{\text{ring properties}\} \\
 & b a_0 + b - b a_k b \\
= & \quad \{(2) \text{ twice}\} \\
 & 1 + b - 1b \\
= & \quad \{\text{ring properties}\} \\
 & 1 \quad .
\end{array}
$$

(End of Proof.)

<u>Lemma 1</u> All the c_k are distinct.

<u>Proof</u> We observe for any h, k, $0 \le h, k < N$

$$
\begin{array}{ll}
 & c_h = c_k \\
= & \quad \{(4)\} \\
 & a_0 + 1 - a_h b = a_0 + 1 - a_k b \\
= & \quad \{\text{ring properties}\} \\
 & a_h b = a_k b \\
\Rightarrow & \quad \{\text{Leibniz}\} \\
 & a_h b a_0 = a_k b a_0 \\
= & \quad \{(2) \text{ with } k := 0, \text{ twice}\} \\
 & a_h 1 = a_k 1 \\
= & \quad \{\text{ring properties}\} \\
 & a_h = a_k \\
\Rightarrow & \quad \{(3)\} \\
 & h = k \quad .
\end{array}
$$

(End of Proof.)

Lemma 2 Each c_k differs from a_0 .

Proof We observe for any k, $0 \le k < N$

$$\begin{array}{ll} & a_0 \ne c_k \\ = & \{(4)\} \\ & a_0 \ne a_0 + 1 - a_k b \\ = & \{\text{ring properties}\} \\ & a_k b \ne 1 \\ = & \{(0) \text{ has no solutions}\} \\ & \text{true} \end{array}$$

(End of Proof.)

Consider now a_0 and the N values c_k . From (2) and Lemma 0 we conclude that these $N+1$ values are all right inverses of b . From Lemmata 1 and 2 we conclude that these $N+1$ values are all distinct. QED

* * *

The above has been triggered by "A Pigeonhole Proof of Kaplansky's Theorem" by Ira Rosenholtz, published in The American Mathematical Monthly, Volume 99, Number 2, February 1992, p. 132-133 . A copy of Rosenholtz's contorted argument is included. It was a pure coincidence that I encountered this article in the AMM a few hours after I had announced in my course "Mathematical Methodology" that next week I would lecture on the Pigeonhole Principle "because it is so frequently misused". Besides the spurious use

A Pigeonhole Proof of Kaplansky's Theorem

Ira Rosenholtz

The purpose of this little note is to sketch a simple proof of the following result, which Kaplansky has referred to as his "infamous little exercise"*. (See [1], [2], [3], [4].)

Theorem (Kaplansky). *Suppose that an element in a ring with identity has two right inverses. Then it has infinitely many right inverses.*

The proof consists of the following two lemmas. It is analogous to solving linear differential equations and is a nice application of the pigeonhole principle.

Lemma 1 (The Homogeneous Solution). *If b has N right inverses with N at least 2, then the equation $bx = 0$ has at least $(N + 1)$ solutions.*

Proof of Lemma 1: Suppose b has distinct right inverses $a_1, a_2, \ldots, a_N$. Then $a_1 - a_1, a_2 - a_1, \ldots, a_N - a_1$ are N distinct solutions of $bx = 0$. We will show that the set $\{1 - a_1 b, 1 - a_2 b, \ldots, 1 - a_N b\}$ contains at least one additional solution of $bx = 0$.

Clearly all of the elements of this set are solutions. If there were not a new solution in this set, then for each j there is a k so that $1 - a_j b = a_k - a_1$. However, $1 - a_j b$ cannot equal $a_1 - a_1 = 0$, because then a_j would be a left inverse for b, and in this case it is easy to see that b has only one right inverse, a contradiction. Thus, since there are N $(1 - a_j b)$'s (the pigeons) and only $(N - 1)$ acceptable $(a_k - a_1)$'s (the pigeon-holes), by the pigeon-hole principle we must have that for some $m \neq n$, $1 - a_m b = 1 - a_n b$. But then $a_m b = a_n b$, and multiplying this on the right by a_1, we get $a_m = a_n$, a contradiction.

Lemma 2 (The Non-Homogeneous Solution). *If b has N right inverses with N at least 2, then b has $(N + 1)$ right inverses.*

Proof of Lemma 2: By Lemma 1, $bx = 0$ has $(N + 1)$ distinct solutions $x_1, x_2, \ldots, x_{N+1}$. But then if a_1 is a right inverse of b, then $\{a_1 + x_1, a_1 + x_2, \ldots, a_1 + x_{N+1}\}$ is a set of $(N + 1)$ distinct right inverses of b.

of the Pigeonhole Principle, it contains a nested reductio ad absurdum!

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